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Louis Padalo

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THE TIME DOMAIN INVERSION
OF CONVOLUTION

A THESIS

Presented to

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by

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THE TIME DOMAIN INVERSION
OF CONVOLUTION

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF ILLUSTRATIONS

SUMMARY

The convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\ g(\tau)\ d\tau\ , \tag{S-1}$$

expressing the function $y(t)$ as the convolution of the functions $x(t)$ and $g(t)$, plays an important role in electrical engineering, arising in the theory of linear systems and in statistical communication theory.

In this research only those functions which vanish when their arguments are negative will be considered. The convolution integral becomes

$$y(t) = \int_{0}^{t} x(t-\tau)\ g(\tau)\ d\tau \tag{S-2}$$

with the limits of integration from 0 to t instead of $-\infty$ to ∞ because of this "causality" assumption.

Convolution is considered to have been inverted when (S-2) has been solved for $g(t)$ in terms of operations on the known functions $y(t)$ and $x(t)$.

The standard method of inverting convolution is by the Laplace Transformation of both sides of (S-2) which yields

$$g(t) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}\right\} \tag{S-3}$$

where $\mathcal{L}\{f(t)\}$ denotes the Laplace Transform of $f(t)$ and $\mathcal{L}^{-1}[\mathcal{L}\{f(t)\}] = f(t)$, as usual.

Since the behavior of $f(t)$ for all time, $0 \leq t < \infty$, must be known before $\mathcal{L}\{f(t)\}$ can be computed, the Laplace Transform inversion of convolution is not a time domain technique and, in particular, it is not a real-time technique.

In this research, methods are developed for inverting convolution in the time domain and, in some cases, in real time. Because of the system theory interpretation of (S-2), where the input to a causal, time-invariant system is $x(t)$, the output is $y(t)$ and the impulse response is $g(t)$, a system-theoretic interpretation of the inversion of convolution is presented.

Three distinct methods of inverting convolution are derived and algorithms for their performance are constructed.

In the first method, motivated by the symbol $\otimes$ used to denote the "convolution product" of $x(t)$ and $g(t)$ in (S-2) as

$$y(t) = x(t) \quad \otimes \quad g(t) \qquad \text{(S-3)}$$

the symbol $\overline{||\qquad\qquad}$ is introduced to denote "convolution division" and an algorithm analogous to ordinary long division is presented so that the inversion of (S-3) appears as

$$g(t) = \{x(o)\ \delta(t) + x'(t)\}\ \overline{||\ y'(t)\ } \qquad \text{(S-4)}$$

where $\delta(t)$ is the unit impulse and the primes denote the ordinary derivatives.

The infinite expansion for $g(t)$ which results from the algorithm expressed by (S-4) is shown to converge to the unique solution $g(t)$ and estimates are made of the rate of convergence under suitable restrictions on $y(t)$ and $x(t)$. The algorithm is seen to involve both differentiation and division to invert the integration and multiplication required by convolution. Some analog inversion systems are presented.

The second method provides for the inversion of analytic functions - those having Taylor series expansions valid in some neighborhood of the origin. It is shown that the Taylor coefficients for $g(t)$ about $t = 0$ are given recursively by

$$g^{(n)}(o) = \frac{1}{x(o)}\left\{y^{(n+1)}(o) - \sum_{j=1}^{n} x^{(j)}(o)\, g^{(n-j)}(o)\right\} \qquad \text{(S-5)}$$

for $n = 1,2, \ldots$

and

$$g(o) = \frac{1}{x(o)}\; y'(o)\ . \qquad \text{(S-6)}$$

Interpreting the Taylor series of a function as the decomposition of the function into the simple unit-singularity functions, the step, the ramp, the parabola, etc., an algorithm analogous to factoring is presented. To illustrate, (S-6) is interpreted as saying that the ramp $y'(o)$ present in $y(t)$ was produced by the step $x(o)$ in $x(t)$ convolved with the step $g(o)$ in $g(t)$. Thus, the amount of step in $g(t)$ is obtained from (S-6).

In the third method the real-time inversion of convolution is accomplished by approximating the waveforms $x(t)$ and $g(t)$ with sufficiently simple functions so that the integration in (S-2) simplifies. The approximating system is then solved for the approximation to $g(t)$.

For example, if $x(t)$ and $g(t)$ are both approximated by rectangular pulses of uniform spacing, h, the convolution in (S-2) appears as

$$y([n+1]h) = h \sum_{j=o}^{n} x(jh)\ g([n-j]h) \qquad \text{(S-7)}$$

with $y(o) = 0$, for $n = 0, 1, 2, 3, \ldots$

The algebraic system of (S-7) is easily solved, recursively, for the rectangular approximation to $g(t)$ yielding

$$g(nh) = \frac{1}{x(0)} \left[\frac{y([n+1]h)}{h} - \sum_{j=o}^{n-1} g(jh)\ x([n-j]h) \right] . \qquad \text{(S-8)}$$

Several other approximating systems, all conceptually as simple as (S-7), are presented and solved with the aid of some results from the algorithm just discussed for analytic functions. It is shown that this third method in its simplest form, (S-7) and (S-8), systematizes and generalizes an easy-to-visualize, trial-and-error method of graphical deconvolution credited to Goodman and Reswick. This interpretation provides the algorithm for the real-time inversion methods.

The real-time methods are easily implemented on the digital computer and provide a practical means for the real-time identification of stationary linear systems by observing the input and output under normal operation.

For those problems where the functions $x(t)$ and $y(t)$ are expressed in terms of the elementary functions, the algorithm of the first method is straightforward. Often for these problems, the algorithm (S-4) simplifies greatly to a few steps instead of requiring an infinite expansion. This is analogous to "cancellation" or "dividing exactly" in ordinary long division and reflects the simple relationship which exists between integral equations such as (S-2) and differential equations.

The second method may also simplify in certain problems but is in general unwieldy. Its principal practical application appears to be in providing the necessary estimates of initial condition terms to begin the recursive solutions in the real time method of inversion.

Being time domain techniques, each of these methods is applicable for a wider class of functions than those which have Laplace Transforms.

Examples are presented to illustrate each of the three inversions and a case where they simplify.

Four appendices attempt to unify and coordinate some of the extensive mathematical research related to convolution and its inversion by the methods of this thesis.

CHAPTER I

INTRODUCTION

The operation of convolution assigns to each pair of functions $x(t)$ and $g(t)$ a third function $y(t)$ by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)g(\tau)d\tau \tag{1-1}$$

provided, of course, that $x(t)$ and $g(t)$ are sufficiently restricted* so that the integral in (1-1) exists.

Long the object of study in mathematics, the convolution integral has played an important role in electrical engineering, arising in the theory of linear systems and in statistical communication theory.

If the input to a causal, time-invariant linear system, Figure 1, is $x(t)$, the output is $y(t)$ and the impulse response is $g(t)$, convolution appears as

$$y(t) = \int_{0}^{t} x(t - \tau)g(\tau)d\tau \tag{1-2}$$

or

$$y(t) = \int_{0}^{t} g(t - \tau)x(\tau)d\tau \quad . \tag{1-3}$$

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*See for example Appendix I, Definition (A1-I) and Theorems (A1-VI) (A1-VII), and (A1-VIII) for sufficient conditions for Riemann integrability.

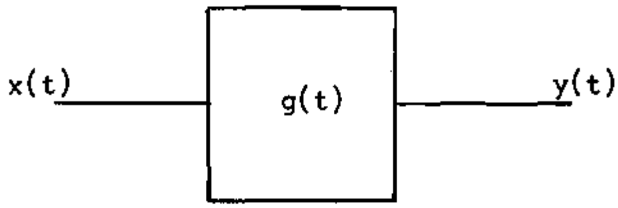

Figure 1. A Causal, Time-Invariant Linear System.

The limits of integration in equations (1-2) and (1-3) are from 0 to t rather than from $-\infty$ to $+\infty$ because of the causality assumptions for both x(t) and g(t).

Convolution appears in statistical communication theory[1] in two contexts:

(i) when x(t) and g(t) are the probability density functions of two statistically independent random signals and y(t) is the density function of the sum of these random signals. (In this case, the functions x(t) and g(t) must be non-negative, as must y(t), for the probabilistic interpretation.)

(ii) when x(t) is the auto-correlation function of the input, y(t) is the cross-correlation function of the output and input, and g(t) is the impulse response of a linear, time-invariant system. (Here x(t) must be an even function.)

For the purpose of this research only those functions x(t) and g(t) which vanish when their arguments are negative, i.e., causal functions, will be considered and the convolution of x(t) and g(t) will hereafter be understood to be defined by Equation (1-2).

Performing Convolution

The sequence of operations called for in (1-2) lends itself to the simple graphical interpretation of Figure 2, where $x(\tau)$, (a); is shifted, (b); folded, (c); and superimposed on the graph of $g(\tau)$, (d).

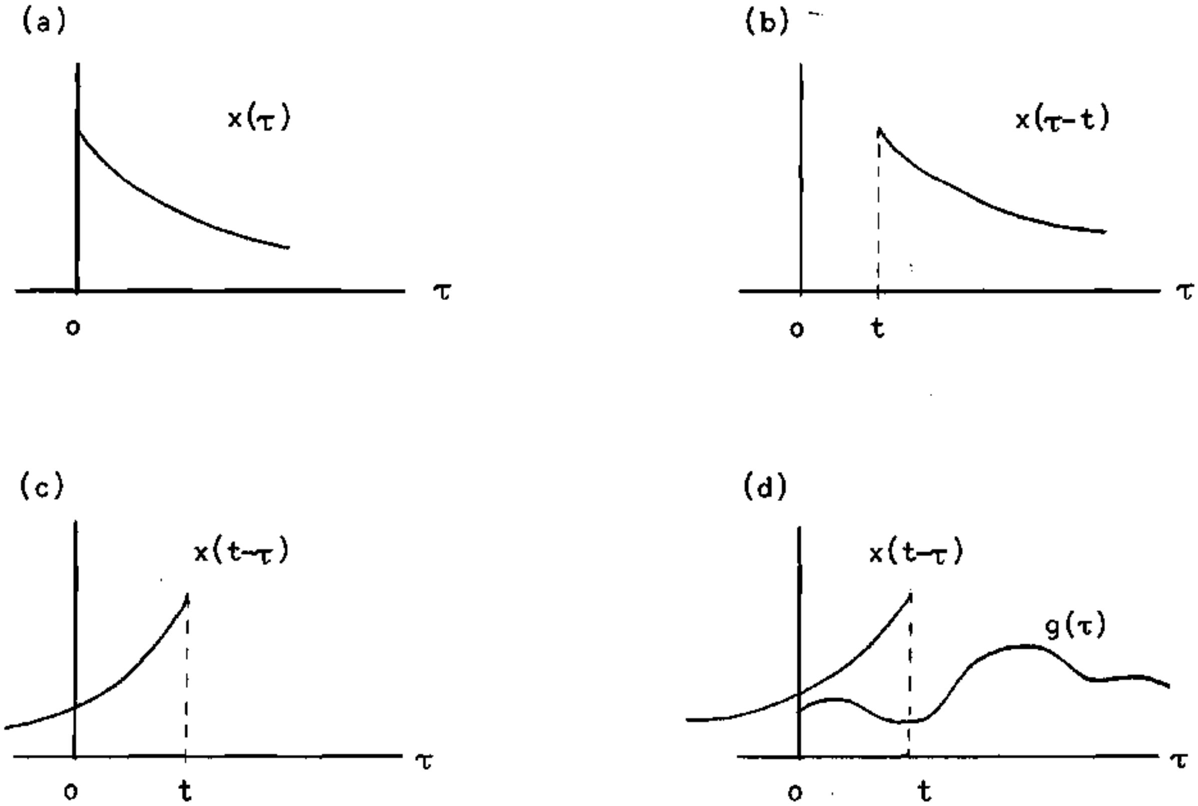

Figure 2. Convolving $x(t)$ and $g(t)$.

The value for $y(t)$ computed by (1-2) then, is the area under the product of the two waveforms of Figure 2d.

The Standard Form for $x(t)$

For the duration of this research, it will be understood that $x(t)$ is in "the standard form" when $x(o)$ is defined and

$$x(o) \neq 0. \tag{1-4}$$

Since

$$x(t) = 0 \tag{1-5}$$

for

$$t < 0 ,$$

condition (1-4) requires that x(t) undergo a step discontinuity from the left at t = 0.

Using the standard notation

$$x^{(n)}(t) \triangleq \frac{d^n}{dt^n} x(t) \tag{1-6}$$

and

$$x^{(-m)}(t) \triangleq \underbrace{\int_o^t \int_o^t \cdots \int_o^t}_{m \text{ times}} x(t) \underbrace{dt\, dt \ldots dt}_{m \text{ times}} , \tag{1-7}$$

if a given x(t) is not already in the standard form (1-4), it is understood that both sides of (1-2) will either be differentiated or integrated the necessary number of times until

$$x^{(n)}(o) \neq 0$$

for some positive or negative integer n.

To illustrate the procedure, suppose x(o) = 0, but that x'(o) ≠ 0 as, for example, in the unit ramp in Figure 3-a.

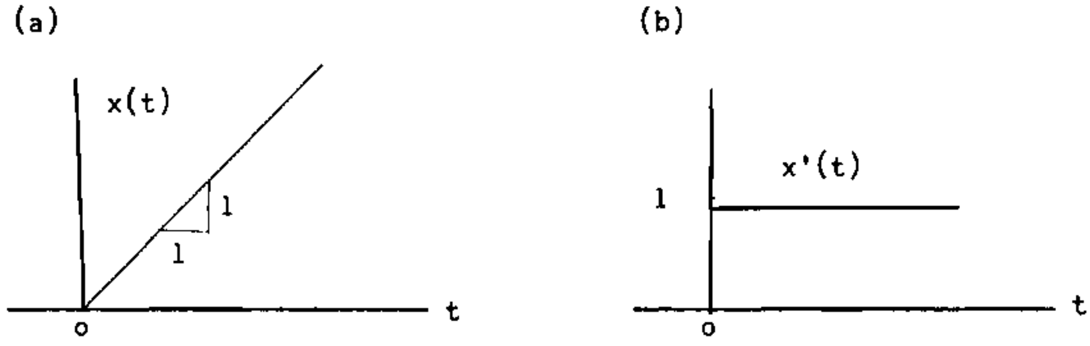

Figure 3. Putting the Unit Ramp Into Standard Form.

Differentiating both sides of (1-2) using Leibniz's rule*, yields

$$y'(t) = \int_0^t x'(t - \tau)g(\tau)d\tau \tag{1-8}$$

where now, $x'(o) = 1 \neq 0$, as in Figure 3b, and (1-8) requires convolving $x'(t)$ and $g(t)$ to produce $y'(t)$. To get $y(t)$, $y'(t)$ is, of course, then integrated.

Similarly, if $x(t)$ contains an impulse at the origin so that $x(o)$ is not defined, a single integration results in a function having the necessary step discontinuity at $t = 0$. Integrating both sides of (1-2) from 0 to t yields

$$\int_0^t y(\tau)d\tau = \int_0^t \int_0^\tau x(\tau - \zeta)g(\zeta)d\zeta \; d\tau \tag{1-9}$$

and interchanging the order of integration

---

*See Appendix I, Theorem (A1-II).

$$\int_0^t y(\tau)d\tau = \int_0^t\int_\zeta^t x(\tau - \zeta)\, d\tau\; g(\zeta)d\zeta \ . \tag{1-10}$$

Changing variables in the inner integral,

$$\int_0^t y(\tau)\ d\tau = \int_0^t \left[\int_0^{(t-\zeta)} x(\tau)\ d\tau\right] g(\zeta)d\zeta \ , \tag{1-11}$$

and (1-11) appears as

$$y^{(-1)}(t) = \int_0^t x^{(-1)}(t-\zeta)g(\zeta)d\zeta \tag{1-12}$$

with $x^{(-1)}(t)$ in standard form.

Inversion of Convolution

Both (1-2) and (1-3) provide for the determination of $y(t)$ by performing the convolution of the known functions $x(t)$ and $g(t)$.

In this research the operation of convolution will be considered to have been inverted when (1-2),

$$y(t) = \int_0^t x(t - \tau)g(\tau)d\tau \ ,$$

has been solved for $g(t)$ in terms of operations on the known functions $x(t)$ and $y(t)$.

In the context of system theory, the inversion of convolution corresponds to determining the impulse response of the system which has response y(t) to excitation x(t). This problem is a form of "the identification problem" where a system is identified by observing its input and output. Because of the symmetry of x(t) and g(t) in (1-2) and (1-3), the same analyses apply to the problems of signal identification, or measurement, and compensation where x(t) is sought from knowledge of g(t) and y(t).

Present Method of Inversion

Convolution is customarily undone by the use of Laplace Transforms. Using the standard lower case - upper case notation for a function f(t) and its Laplace Transform F(s)

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt\ , \tag{1-13}$$

and

$$X(s) = \mathcal{L}\{x(t)\} \tag{1-14}$$

$$G(s) = \mathcal{L}\{g(t)\} \tag{1-15}$$

$$Y(s) = \mathcal{L}\{y(t)\}\ . \tag{1-16}$$

Taking Laplace Transforms of both sides of (1-2) yields

$$Y(s) = X(s)G(s) \tag{1-17}$$

because the operation of convolution in the time domain corresponds to multiplication in the Laplace Transform, or frequency, domain.

Thus, the customary frequency domain inversion is

$$g(t) = \mathcal{L}^{-1}\left[\frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}\right] \tag{1-18}$$

where

$$\mathcal{L}^{-1}[\mathcal{L}\{f(t)\}] \triangleq f(t) \tag{1-19}$$

as usual.

The operations of (1-18) are straightforward when $x(t)$ and $y(t)$ are analytically defined, but become quite involved computationally when $x(t)$ and $y(t)$ are given numerically. In addition, as seen by (1-13) it is necessary that the entire waveforms for $x(t)$ and $y(t)$ be known before $g(t)$ can be computed by the frequency domain technique of (1-18). Thus, not only is (1-18) not a time domain technique, it is also not a "real time" technique.

A system-theoretic interpretation of the inversion (1-18) has been given by Turin[2] who derives the transfer function $H(s)$ of an "inverse filter" which when excited by the output $y(t)$ of Figure 1 will yield as its response the desired $g(t)$. This cascaded system is illustrated in Figure 4 in terms of Laplace Transforms and transfer functions.

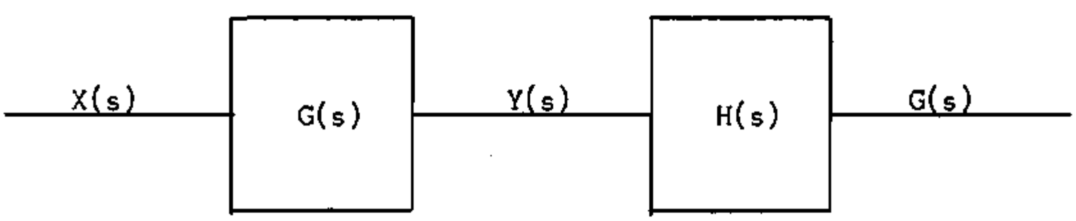

Figure 4. Turin's Cascaded System.

Since

$$Y(s) = X(s)G(s)$$

and

$$G(s) = H(s)Y(s)$$

it follows that

$$H(s) = \frac{1}{X(s)} \quad (1\text{-}20)$$

Interchanging the order of the two systems in Figure 4, Turin gets the cascade of Figure 5

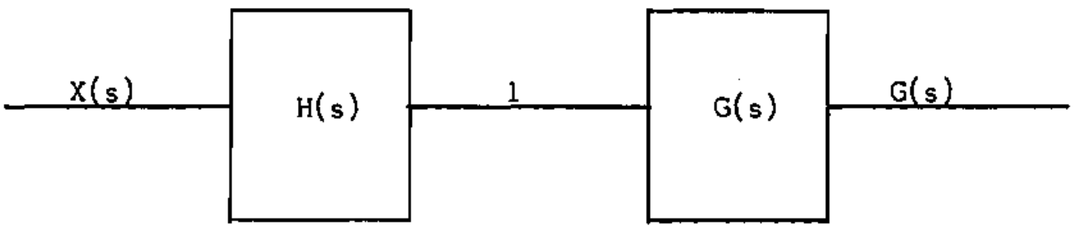

Figure 5. Rearrangement of Figure 4.

which reveals that his filter, H(s) transforms x(t) into the unit impulse $\delta(t)$. With $\delta(t)$ as its input, G(s) of course has its impulse response g(t) as output.

Taking the inverse transform of both sides of (1-20) the impulse response, H(t), of the inverse filter is given by

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{\mathcal{L}\{x(t)\}}\right] \quad (1\text{-}21)$$

and is again a frequency domain technique.

Turin's inverse filter may be interpreted in the time domain by multiplying both sides of (1-20) by X(s) and then inverting to get

$$\delta(t) = \int_0^t x(t - \tau)h(\tau)\,d\tau\,. \qquad (1\text{-}22)$$

This suggests the notion of a pair of functions being inverse with respect to the unit impulse. Turin's h(t) will then be considered as "the convolution inverse of x(t) with respect to the unit impulse."

Zames[3] derives a feedback model of the inverse filter H(s) by the following formal manipulations:

Write

$$H(s) = \frac{1}{X(s)}$$

as

$$H(s) = \left.\frac{a}{(1-ab) + a\,X(s)}\right|_{\text{for } b = \frac{1}{a}}\,. \qquad (1\text{-}23)$$

Using his feedback system implementation of (1-23), the inversion system of Figure 4 appears as shown in Figure 6.

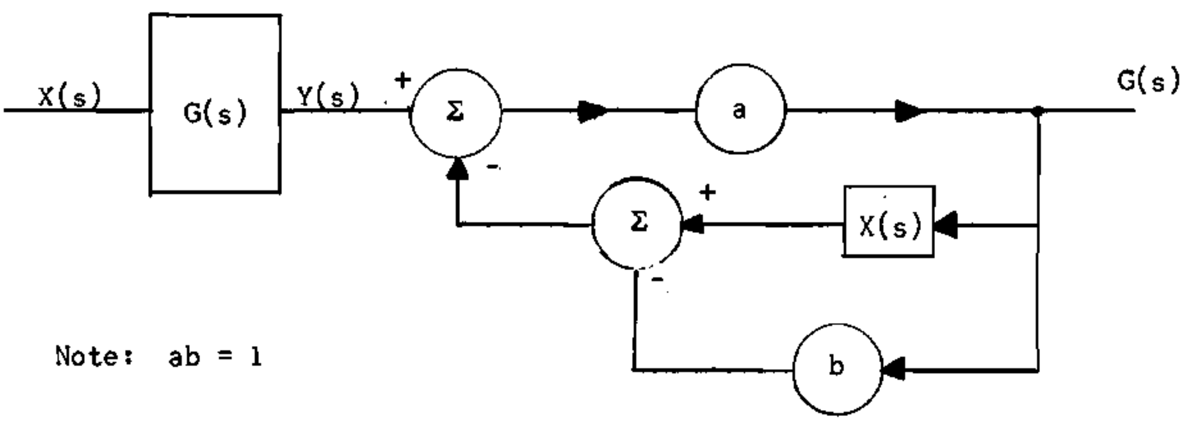

Figure 6. Zames's Feedback Inverse System.

Zames points out that this is only a formal procedure and that the system may not physically realizable. This point will be discussed in more detail in Chapter V.

There do appear in the literature several approaches to the inversion of convolution in the time domain. Briefly, these methods assume some a priori knowledge about the system to be identified such as the order of the describing differential equation, or the number of integrators in a circuit, and then use the input-output data to determine the scale factors and coefficients of the model.

Berman, Weiss and Shahn,[4] Mason and Zimmerman,[5] and Zabusky[6] typify this approach. Mason and Zimmerman[5] present an interesting "partial fraction" expansion of their inverse convolution and introduce a symbol, //, to stand for the inversion.

Lewis,[7] Papoulis[8] and others assume band-limited signals in order to represent waveforms by time series - a method developed by Tustin.[9] Goodman and Hillsley[10] and Papoulis[8] typify the approach where moments of the impulse response are computed - usually under quite severe assumptions.

The term "deconvolution" to represent the inversion process is used in Goodman and Reswick[11] and in Levin.[12]

Objective of the Research

This thesis addresses itself to the problem of inverting convolution in the time domain and wherever possible, in real time. It is hoped that the algorithms presented will provide as much help in visualizing the steps in inverting convolution as the steps of Figure 2 provide in performing it.

Outline of the Research

The problems and results of the research will now be outlined briefly, without detailed proofs, to describe the content of the work.

In Chapter II, an algorithm for the time domain inversion of convolution is developed and shown to be analogous to long division as an inversion of ordinary multiplication. The algorithm is seen to involve differentiation and division - two operations intuitively inverse to the operations of integration and multiplication comprising convolution.

Motivated by the symbol $\otimes$ frequently used to denote the "convolution product"* of two functions $v(t)$ and $u(t)$ as

$$v(t) \otimes u(t) \,,$$

the symbol $\Vert\overline{}$ is introduced to represent the "convolution division" of a function $u(t)$ by $v(t)$ as

$$v(t) \; \Vert\overline{\,u(t)\,} \,.$$

With this notation, Eq. (1-2) appears as

$$y(t) = x(t) \otimes g(t) \tag{1-24}$$

and the algorithm is shown to provide the inversion of (1-24) as

$$g(t) = \{x(o)\,\delta(t) + x'(t)\} \; \Vert\overline{\,y'(t)\,} \,. \tag{1-25}$$

---

*So called because convolution in (1-2) and (1-3) satisfies all the axioms of group multiplication in the algebraic sense. See Mikusinski, Ref. (13).

The steps to be carried out in (1-25) are found to be precisely those of "trial divisor" and "remainder" of ordinary long division.

The resulting expression for $g(t)$ is found to be

$$g(t) = \frac{1}{x(o)} y'(t) - \left[\frac{1}{x(o)}\right]^2 x'(t) \otimes y'(t) + \left[\frac{1}{x(o)}\right]^3 x'(t) \otimes x'(t) \otimes y'(t) + \ldots \tag{1-26}$$

and this infinite expansion is shown to converge uniformly and absolutely to the unique solution of (1-2) under suitable restrictions on $x(t)$ and $y(t)$.

Inverting convolution by (1-25) is then shown to be equivalent to finding a function $z(t)$ which is "the convolution inverse of $\frac{1}{x(o)} x(t)$ with respect to the unit step." That is,

$$1 = \int_o^t z(t - \tau) \frac{x(\tau)}{x(o)} d\tau \ . \tag{1-27}$$

This suggests a system theory approach similar to Turin's and Zames's, and block diagrams of some analog inversion systems are presented.

In Chapter III, motivated by the infinite expansions of the algorithm of Chapter II, an inversion process is developed for analytic functions. By repeated applications of Leibniz's rule to Eq. (1-2) the n^{th} derivative of $g(t)$ is shown to be

$$g^{(n)}(t) = \frac{1}{x(o)} \left[y^{(n+1)}(t) - \int_o^t x^{(n+1)}(t-\tau)\, g(\tau)\, d\tau - \sum_{j=1}^{n} x^{(j)}(o)\, g^{(n-j)}(t) \right]. \tag{1-28}$$

Letting $t = 0$ in (1-28), assuming no impulses in the integrand,

$$g^{(n)}(o) = \frac{1}{x(o)} \left[y^{(n+1)}(o) - \sum_{j=1}^{n} x^{(j)}(o)\, g^{(n-j)}(o) \right] \tag{1-29}$$

for $n = 1, 2, \ldots$

with

$$g(o) = \frac{1}{x(o)}\, y'(o)\ . \tag{1-30}$$

Thus the Taylor expansion of $g(t)$ about the origin is determined by the derivatives of the analytic functions $x(t)$ and $y(t)$.

The same results, (1-29) and (1-30) are obtained by convolving the Taylor expansions of $x(t)$ and $g(t)$ to get the Taylor expansion for $y(t)$ from

$$y^{(k+1)}(o) = \sum_{j=o}^{k} g^{(j)}(o)\, x^{(k-j)}(o) \tag{1-31}$$

for $k = 0, 1, 2, \ldots$

It is observed that (1-31) involves a "convolution summation" on the orders of the derivatives of $x(t)$ and $g(t)$.

Since the Taylor expansion of a function permits the visualization of decomposing a function into the unit singularity functions, impulses, steps, ramps, parabolas, etc., the results of Chapter III are also amenable to a graphical or algorithmic interpretation. To illustrate, if $y(o)$ is zero in (1-2) and $x(o)$ is finite, there can be no impulse in $g(t)$ at the origin; for, any step in $y(t)$, $y(o)$, must be due to impulse of strength $y(o)/x(o)$ at the origin in $g(t)$. Similarly, assuming no impulse in $g(t)$, any ramp in $y(t)$, $y'(o)$, is caused by the step in $x(t)$, $x(o)$, convolving with the step in $g(t)$, $g(o)$. Thus

$$g(o)\, x(o) = y'(o)$$

which is just (1-30).

While lacking the elegance of the algorithm of Chapter II the methods of Chapter IV possess a certain appeal of their own resembling the method of factoring before multiplication and keeping track of the factors. In addition, they are vital to the practical, real-time methods of Chapter IV, providing estimates of initial condition terms needed to begin certain recursive solutions of difference equations.

In Chapter IV practical, real-time methods of inverting convolution are developed and discussed. All of these methods involve approximating the waveforms $x(t)$ and $g(t)$ by a collection of sufficiently simple functions so that the integration of (1-2) simplifies to a sum. The resulting algebraic system of linear equations is then solved for the approximation to $g(t)$.

The methods of Chapter IV are shown to generalize and systematize an easy-to-visualize, trial-and-error, graphical method credited to

Goodman and Reswick.[11]

Chapter V presents examples which illustrate each of the three inversions and a case where they simplify.

Four appendices attempt to coordinate some of the extensive mathematical literature relevant to convolution and its inversion by the methods of this research. The appendices also contain some derivations too lengthy to include in the main text of the thesis.

CHAPTER II

A DIVISION ALGORITHM FOR THE INVERSION OF CONVOLUTION

Development of the Algorithm

Consider

$$y(t) = \int_o^t x(t-\tau)\ g(\tau)\ d\tau \tag{2-1}$$

written in the equivalent form

$$y(t) = x(o) \int_o^t g(\tau)\ d\tau + \int_o^t [x(t-\tau) - x(o)]\ g(\tau)\ d\tau\ . \tag{2-2}$$

Taking the Laplace Transform of both sides,

$$Y(s) = \frac{x(o)\ G(s)}{s} + X(s)\ G(s) - \frac{x(o)\ G(s)}{s}\ , \tag{2-3}$$

whence

$$Y(s) = \frac{x(o)\ G(s)}{s} + [X(s) - \frac{x(o)}{s}]\ G(s)\ , \tag{2-4}$$

or

$$G(s) = \frac{sY(s)}{x(o)} - \frac{[sX(s) - x(o)]}{x(o)}\ G(s)\ , \tag{2-5}$$

since $x(o) \neq 0$.*

---

*$x(t)$ is in "standard form." See Chapter I, page 3.

This may be written

$$G(s) = F(s) - \left[\frac{1}{x(o)}\right] K(s)\, G(s) \tag{2-6}$$

where

$$F(s) \triangleq \frac{sY(s)}{x(o)} \tag{2-7}$$

and

$$K(s) \triangleq sX(s) - x(o) \ . \tag{2-8}$$

Clearly, since $y(o) = 0$,

$$f(t) \triangleq \mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{x(o)}\, y'(t) \tag{2-9}$$

and

$$k(t) \triangleq \mathcal{L}^{-1}\left\{K(s)\right\} = x'(t). \tag{2-10}$$

With this notation, then,

$$G(s) = \frac{1}{1 + \frac{1}{x(o)} K(s)}\, F(s). \tag{2-11}$$

Now

$$\frac{1}{1 + \frac{1}{x(o)} K(s)} = 1 - \frac{1}{x(o)} K(s) + \left[\frac{1}{x(o)}\right]^2 \left\{K(s)\right\}^2 + \ldots + (-1)^n \left[\frac{1}{x(o)}\right]^n \left\{K(s)\right\}^n + \ldots \tag{2-12}$$

provided

$$|K(s)| < |x(o)| .$$

By virtue of this expansion

$$G(s) = \left[1 - \frac{1}{x(o)} K(s) + \left(\frac{1}{x(o)}\right)^2 \{K(s)\}^2 + \ldots + (-1)^n \left(\frac{1}{x(o)}\right)^n K(s)^{\;n} + \ldots\right] F(s) \tag{2-13}$$

and, equivalently,

$$G(s) = F(s) - \frac{1}{x(o)} K(s) F(s) + \left[\frac{1}{x(o)}\right]^2 \{K(s)\}^2 F(s) + \ldots + (-1)^n \left[\frac{1}{x(o)}\right]^n \{K(s)\}^n F(s) + \ldots \tag{2-14}$$

Consider, first, the expression for $G(s)$ in (2-14).

Introducing the notation

$$k_n(t) \triangleq \mathcal{L}^{-1}\left[\{K(s)\}^n\right], \quad \text{for} \quad n \geq 1, \tag{2-15}$$

where

$$k_1(t) \triangleq k(t), \tag{2-16}$$

it follows that the function $k_n(t)$ is the n-fold, self-convolution of $k(t)$.

Each term in the geometric series of (2-14) is a product of two Laplace Transforms and thus corresponds to the convolution of the corresponding functions in the time domain.

Taking the inverse Laplace Transform of both sides of (2-14) the following serial expression for $g(t)$ results:

$$g(t) = f(t) - \frac{1}{x(o)} \int_o^t k(t - \tau)\ f(\tau)\ d\tau +$$

$$\left[\frac{1}{x(o)}\right]^2 \int_o^t k_2(t - \tau)\ f(\tau)\ d\tau + \ldots + \tag{2-17}$$

$$(-1)^n \left[\frac{1}{x(o)}\right]^n \int_o^t k_n(t - \tau)\ f(\tau)\ d\tau + \ldots$$

Recalling that

$$f(t) = \frac{1}{x(o)}\, y'(t)$$

and

$$k(t) = x'(t)$$

$$k_2(t) = \int_o^t x'(t - \tau)\ x'(\tau)\ d\tau$$

$$k_3(t) = \int_o^t x'(t - \tau)\ k_2(\tau)\ d\tau \tag{2-18}$$

$$k_n(t) = \int_o^t x'(t - \tau)\ k_{n-1}(\tau)\ d\tau$$

for $n = 2, 3, \ldots,$

the inversion algorithm of Eq. (2-17) is seen to be analogous to the long division algorithm for inverting ordinary multiplication.

Each of the terms of (2-17) may be thought of as representing a "partial quotient" or a "trial divisor." In fact, (2-17) suggests the formal symbolism:

$$
\begin{array}{r|l}
 & f(t) - \frac{1}{x(o)} k(t) \otimes f(t) + \ldots \\
\hline
\left\{\delta(t) + \frac{1}{x(o)} k(t)\right\} & f(t) \\
 & \underline{f(t) + \frac{1}{x(o)} k(t) \otimes f(t)} \\
 & - \frac{1}{x(o)} k(t) \otimes f(t) \\
 & \underline{- \frac{1}{x(o)} k(t) \otimes f(t) - \left[\frac{1}{x(o)}\right]^2 k_2(t) \otimes f(t)} \\
 & \left[\frac{1}{x(o)}\right]^2 k_2(t) \otimes f(t) \\
 & \vdots
\end{array}
\qquad (2.19)
$$

where the symbol $\otimes$ denotes convolution multiplication and $q(t)\,\overline{\|\; p(t)}$ denotes the "convolution division" of $p(t)$ by $q(t)$.

Thus,

$$
g(t) = \left\{x(o)\,\delta(t) + x'(t)\right\} \overline{\|\; y'(t)} \qquad (2\text{-}20)
$$

as illustrated in (2-19), is the inversion algorithm for the convolution product of Eq. (2-1).

Analysis of the Algorithm

Consider now Eq. (2-11).

Letting

$$
Z(s) \triangleq \frac{1}{1 + \frac{1}{x(o)} K(s)} \qquad (2\text{-}21)
$$

it is seen from (2-13) that

$$
Z(s) = 1 - \frac{1}{x(o)} K(s) + \ldots + (-1)^n \left[\frac{1}{x(o)}\right]^n \left\{K(s)\right\}^n + \ldots \qquad (2\text{-}22)
$$

whence, if

$$z(t) = \mathcal{L}^{-1}\{Z(s)\} , \tag{2-23}$$

$$z(t) = \delta(t) - \frac{1}{x(o)} k(t) + \ldots + (-1)^n \left[\frac{1}{x(o)}\right]^n k_n(t) + \ldots \tag{2-24}$$

and the solution $g(t)$ is given by

$$g(t) = \int_o^t z(t - \tau) \; f(\tau) \; d\tau \tag{2-25}$$

or

$$g(t) = \frac{1}{x(o)} \int_o^t z(t - \tau) \; y'(\tau) \; d\tau. \tag{2-26}$$

Substituting (2-26) for $g(t)$ in (2-1),

$$y(t) = \int_o^t x(t - \tau) \left[\frac{1}{x(o)} \int_o^\tau z(\tau - \zeta) \; y'(\zeta) \; d\zeta\right] d\tau . \tag{2-27}$$

When the order of the integration is changed,

$$y(t) = \int_o^t \left[\frac{1}{x(o)} \int_\zeta^t x(t - \tau) \; z(\tau - \zeta) \; d\tau\right] y'(\zeta) \; d\zeta . \tag{2-28}$$

Since

$$y(t) = \int_o^t y'(\zeta) \; d\zeta , \tag{2-29}$$

the equality

$$\int_0^t y'(\zeta)\, d\zeta = \int_0^t \left[\frac{1}{x(o)} \int_\zeta^t x(t - \tau)\; z(\tau - \zeta)\; d\tau\right] y'(\zeta)\, d\zeta \qquad \text{(2-30)}$$

requires that

$$1 = \frac{1}{x(o)} \int_\zeta^t x(t - \tau)\; z(\tau - \zeta)\; d\tau\, . \qquad \text{(2-31)}$$

Changing variables, (2-31) becomes

$$1 = \frac{1}{x(o)} \int_0^{t-\zeta} x(t - \zeta - \tau)\; z(\tau)\; d\tau \qquad \text{(2-32)}$$

or

$$1 = \int_0^t x(t - \tau)\, \frac{z(\tau)}{x(o)}\, d\tau\, . \qquad \text{(2-33)}$$

Eq. (2-33) shows that $z(t)$ convolves with $x(t)$ to give a step function. In this sense, $z(t)$ is the "convolution inverse of $\frac{1}{x(o)}\, x(t)$ with respect to the unit step."

From (2-26) it is clear that $\frac{1}{x(o)}\, z(t)$, where $z(t)$ is determined from $x(t)$ by Eq. (2-24), is the impulse response of the filter which when driven by $y'(t)$, yields $g(t)$ as its response.

Conceptually, this is portrayed in the block diagram of Figure 7.

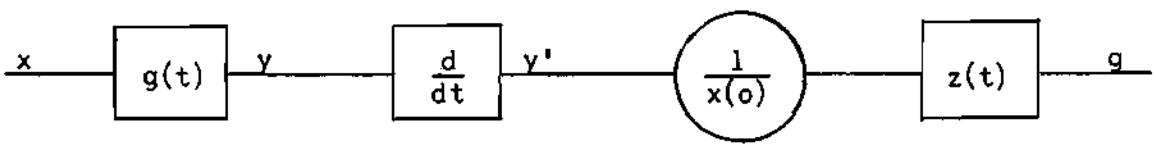

Figure 7. Block Diagram for Equation (2-26).

From Eq. (2-11), however, the "feedback system" analog of Figure 8, is suggested and should be compared with Zames's feedback system in Figure 6.

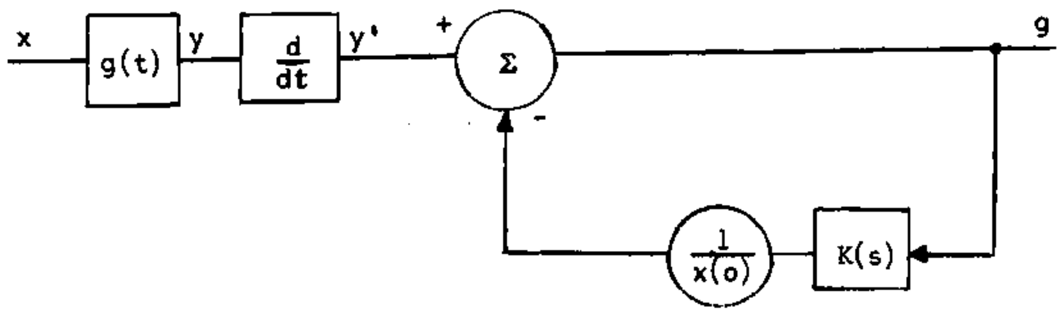

Figure 8. Feedback System Equivalent of Figure 7.

Establishing the convergence of the infinite expansions of Equations (2-12), (2-17), (2-22), and (2-24), then, is seen to be equivalent to establishing the stability of (2-21) in its "feedback system" counterpart in Figure 7.

The relationship between Turin's inverse filter's impulse response $h(t)$ and this function $z(t)$ is easily seen by taking Laplace Transforms in (2-33) to get

$$\frac{1}{s} = \frac{X(s)}{x(o)} \, Z(s) \, .$$

Thus

$$Z(s) = \frac{x(o)}{s} \frac{1}{X(s)} \qquad (2\text{-}34)$$

or

$$Z(s) = \frac{x(o)}{s} H(s) \tag{2-35}$$

from equation (1-20).

In the time domain (2-35) becomes

$$z(t) = x(o) \int_o^t h(\tau)\, d\tau \tag{2-36}$$

and, equivalently,

$$h(t) = \frac{1}{x(o)} \; z'(t) \; . \tag{2-37}$$

Equations (2-35), (2-36), and (2-37) provide the formal relationships required to get the equivalent Turin system, Figure 4 from the inversion systems of Figure 7 and Figure 8.

It is observed from (2-37) that $h(t)$ will contain a unit doublet component because of the differentiation of $z(t)$ which, from (2-24), contains a unit impulse. Since convolving a function with a unit doublet is operationally equivalent to differentiating the function, and since $h(t)$ contains a unit doublet, it is seen that any inversion system designed to produce $g(t)$ from $y(t)$ must necessarily involve differentiation.

This fact is revealed explicitly in equation (2-26) and the block diagrams of Figure 7 and Figure 8.

In order to deal only with ordinary functions rather than generalized functions containing impulses and doublets, it will be convenient at this stage to define the function $r(t)$ by

$$r(t) = k(t) - \frac{1}{x(o)} k_2(t) + \left[\frac{1}{x(o)}\right]^2 k_3(t) + \ldots + (-1)^n \left[\frac{1}{x(o)}\right]^n k_{n+1}(t) + \ldots \qquad (2\text{-}38)$$

This function $r(t)$ is seen from (2-24) to be the "non-impulsive" part of $z(t)$,

$$z(t) = \delta(t) - \frac{1}{x(o)} r(t) \,. \qquad (2\text{-}39)$$

Taking Laplace Transforms,

$$Z(s) = 1 - \frac{1}{x(o)} R(s) \qquad (2\text{-}40)$$

and comparison with (2-21) yields the reciprocal relationship of $K(s)$ and $R(s)$ as

$$\frac{1}{1 + \frac{1}{x(o)} K(s)} = 1 - \frac{1}{x(o)} R(s) \,. \qquad (2\text{-}41)$$

Rationalizing (2-41)

$$1 = \left[1 + \frac{1}{x(o)} K(s)\right] \left[1 - \frac{1}{x(o)} R(s)\right] \qquad (2\text{-}42)$$

or

$$1 = 1 + \frac{1}{x(o)} K(s) - \frac{1}{x(o)} R(s) - \left[\frac{1}{x(o)}\right]^2 K(s)\, R(s).$$

Thus

$$K(s) = R(s) + \frac{1}{x(o)} K(s)\, R(s) \qquad (2\text{-}43)$$

or

$$R(s) = K(s) - \frac{1}{x(o)} R(s)\ K(s)\ . \tag{2-44}$$

These may also be written

$$K(s) = \frac{R(s)}{1 - \frac{1}{x(o)} R(s)} \tag{2-45}$$

and

$$R(s) = \frac{K(s)}{1 + \frac{1}{x(o)} K(s)} \tag{2-46}$$

to further display the reciprocity between $R(s)$ and $K(s)$.

Taking inverse Laplace Transforms of (2-43) and (2-44), the time domain reciprocity appears as

$$k(t) = r(t) + \frac{1}{x(o)} \int_o^t r(t - \tau)\ k(\tau)\ d\tau \tag{2-47}$$

and

$$r(t) = k(t) - \frac{1}{x(o)} \int_o^t k(t - \tau)\ r(\tau)\ d\tau\ . \tag{2-48}$$

This last result will be used to show that the infinite expansion of (2-38) does indeed provide the solution of (2-1).

Convergence of the Algorithm

It will now be shown that the infinite expansion of (2-17) converges uniformly and absolutely to the unique continuous solution of (2-1) if the functions $x(t)$ and $y(t)$ are continuously differentiable.*

Consider the infinite series

$$r(t) = k(t) - \frac{1}{x(o)}\, k_2(t) + \left[\frac{1}{x(o)}\right]^2 k_3(t) + \cdots + (-1)^n \left[\frac{1}{x(o)}\right]^n k_{n+1}(t) \cdots \tag{2-49}$$

The assumed continuity of $x'(t)$ and $y'(t)$ on the interval $[o,b]$ implies that $k(t)$ and $f(t)$ are bounded, i.e. there exists numbers $m \geq 0$, $M \geq 0$ such that

$$|f(t)| \leq m \quad \text{and} \quad |k(t)| \leq M \quad \text{for} \quad t \in [o,b] \, . \tag{2-50}$$

Recall

$$k_2(t) = \int_o^t k(t-\tau)\, k(\tau)\, d\tau$$

so

$$|k_2(t)| \leq \int_o^t |k(t-\tau)|\, |k(\tau)|\, d\tau \leq M^2 t \, . \tag{2-51}$$

Similarly

$$|k_3(t)| \leq \int_o^t |k(t-\tau)|\, |k_2(\tau)|\, d\tau \leq M^3 \frac{t^2}{2} \tag{2-52}$$

and

---

*See Appendix III for some less restrictive conditions sufficient for convergence.

$$|k_4(t)| \leq \int_o^t |k(t-\tau)| \; |k_2(\tau)| \; d\tau \leq M^4 \frac{t^3}{3!} \; . \tag{2-53}$$

The pattern indicates that

$$|k_n(t)| \leq M^n \frac{t^{n-1}}{(n-1)!} \tag{2-54}$$

for $n \geq 1$, and an easy proof by induction establishes the validity of (2-54).

Each term of the series (2-49) is thus dominated by the corresponding term of a series known to converge uniformly and absolutely. That is, since

$$\left| (-1)^n \left[\frac{1}{x(o)}\right]^n k_{n+1}(t) \right| \leq \left| (-1)^n \left[\frac{1}{x(o)}\right]^n \right| M^{n+1} \frac{t^n}{n!} \tag{2-55}$$

and since

$$\sum_{n=o}^{\infty} \left| - \frac{1}{x(o)} \right|^n \frac{M^n}{n!} t^n = e^{\left|\frac{M}{x(o)}\right| t} , \tag{2-56}$$

uniformly, for all $t \geq 0$,

then

$$r(t) = \sum_{n=o}^{\infty} \left[\frac{-1}{x(o)}\right]^n k_{n+1}(t) \tag{2-57}$$

converges uniformly and absolutely and, by the assumed continuity of $k(t)$, is continuous on $[o,b]$.

Moreover,

$$|r(t)| \leq \sum_{n=o}^{\infty} \left|\frac{-1}{x(o)}\right|^n \left|k_{n+1}(t)\right| \leq M\, e^{\left|\frac{M}{x(o)}\right| t} \tag{2-58}$$

for all $t \geq 0$,

and

$$|r(t)| \leq M\, e^{\left|\frac{M}{x(o)}\right| b} \tag{2-59}$$

for all $t \varepsilon [o,b]$ where b is the largest t of interest.

The uniform convergence of $r(t)$ insures that its infinite series may be integrated term by term, as called for in (2-26), (2-47) and (2-48).

Verification of the Inversion

The series (2-57) for $r(t)$ will now be shown to satisfy (2-48). Consider

$$\int_o^t k(t - \tau)\, r(\tau)\, d\tau \; .$$

By (2-57)

$$\int_o^t k(t - \tau)\, r(\tau)\, d\tau = \int_o^t k(t - \tau) \left[\sum_{n=o}^{\infty} \left(- \frac{1}{x(o)}\right)^n k_{n+1}(\tau) \right] d\tau \tag{2-60}$$

and, by uniformity,

$$\int_o^t k(t - \tau)\, r(\tau)\, d\tau = \sum_{n=o}^{\infty} \left(- \frac{1}{x(o)}\right)^n \int_o^t k(t - \tau)\, k_{n+1}(\tau)\, d\tau . \tag{2-61}$$

By (2-18)

$$\int_o^t k(t - \tau)\ k_{n+1}\ (\tau)\ d\tau = k_{n+2}(\tau) \tag{2-62}$$

so that

$$\int_o^t k(t - \tau)\ r(\tau)\ d\tau = \sum_{n=o}^{\infty} \left[-\frac{1}{x(o)}\right]^n k_{n+2}(t)$$

$$= k_2(t) - \frac{1}{x(o)} k_3(t) + \left[\frac{1}{x(o)}\right]^2 k_4(t) + \ldots \tag{2-63}$$

Thus

$$k(t) - \frac{1}{x(o)} \int_o^t k(t - \tau)\ r(\tau)\ d\tau =$$

$$k(t) - \frac{1}{x(o)} k_2(t) + \left[\frac{1}{x(o)}\right]^2 k_3(t) + \ldots \tag{2-64}$$

The right hand side of (2-64) is precisely the series (2-57) for $r(t)$, so

$$k(t) - \frac{1}{x(o)} \int_o^t k(t - \tau)\ r(\tau)\ d\tau = r(t)$$

and it is verified that the infinite expansion designated as $r(t)$ by (2-57) satisfies (2-48).

Since this construction for $r(t)$ satisfies (2-48) and since all the operations leading from (2-1) to (2-48) are reversible, it follows that this $r(t)$ provides the solution of (2-1).

In fact, from (2-39) and (2-26)

$$z(t) = \delta(t) - \frac{1}{x(o)} r(t)$$

and

$$g(t) = \frac{1}{x(o)} \int_o^t \left[\delta(t - \tau) - \frac{1}{x(o)} r(t - \tau)\right] y'(\tau)\, d\tau \tag{2-65}$$

whence the solution is

$$g(t) = \frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_o^t r(t - \tau)\, y'(\tau)\, d\tau\right]. \tag{2-66}$$

That (2-66) is the solution may also be verified directly in the time domain for then

$$\int_o^t k(t - \tau)\, g(\tau)\, d\tau =$$

$$\int_o^t k(t - \tau) \left[\frac{1}{x(o)} \left(y'(\tau) - \frac{1}{x(o)} \int_o^\tau r(\tau - \zeta) y'(\zeta)\, d\zeta\right)\right] d\tau$$

$$= \frac{1}{x(o)} \int_o^t k(t-\tau)\, y'(\tau)\, d\tau + \frac{1}{x(o)} \int_o^t \left[- \frac{1}{x(o)} \int_\zeta^t k(t-\tau)\, r(t-\zeta)\, d\tau\right] y'(\zeta)\, d\zeta$$

which by (2-48), yields

$$\int_o^t k(t-\tau)\, g(\tau)\, d\tau =$$

$$\frac{1}{x(o)} \int_o^t k(t-\tau)\, y'(\tau)\, d\tau + \frac{1}{x(o)} \int_o^t \left[r(t-\zeta) - k(t-\zeta)\right] y'(\zeta)\, d\zeta$$

or

$$\int_0^t k(t-\tau)\, g(\tau)\, d\tau = \frac{1}{x(o)} \int_0^t r(t-\zeta)\, y'(\zeta)\, d\zeta \, . \qquad (2\text{-}67)$$

Thus

$$y'(t) - \frac{1}{x(o)} \int_0^t r(t-\zeta)\, y'(\zeta)\, d\zeta = y'(t) - \int_0^t k(t-\tau)\, g(\tau)\, d\tau$$

and

$$\frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_0^t r(t-\zeta)\, y'(\zeta)\, d\zeta \right] =$$

$$\frac{y'(t)}{x(o)} - \frac{1}{x(o)} \int_0^t x'(t-\tau)\, g(\tau)\, d\tau \, . \qquad (2\text{-}68)$$

That the right hand side of (2-68) is $g(t)$ is seen by differentiating both sides of Eq. (2-2)

$$y'(\tau) = x(o)\, g(t) + \int_0^t x'(t-\tau)\, g(\tau)\, d\tau \, . \qquad (2\text{-}69)$$

Hence

$$\frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_0^t r(t-\tau)\, y'(\tau)\, d\tau \right] = g(t)$$

and (2-66) is verified.

Uniqueness

It remains to show that $r(t)$ is unique.

Suppose there were two functions $r_1(t)$ and $r_2(t)$ such that (2-66) was satisfied.

Then

$$g(t) = \frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_o^t r_1(t-\tau)\; y'(\tau)\; d\tau \right]$$

and

$$g(t) = \frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_o^t r_2(t-\tau)\; y'(\tau)\; d\tau \right] .$$

Subtracting,

$$0 = \int_o^t \left[r_1(t-\tau) - r_2(t-\tau) \right] y'(\tau)\; d\tau \; .$$

Being the difference of two continuous functions, $[r_1(t-\tau) - r_2(t-\tau)]$ is continuous and the assumed continuity of the arbitrary function $y'(\tau)$ shows that

$$r_1(t) \equiv r_2(t)$$

and $r(t)$ is unique.

Rate of Convergence and Error Bounds

The infinite expansion of (2-17) may be identified with the Neumann series solution of the Volterra integral equation of the second kind

$$\varphi(t) - \lambda \int_a^t K(t,\tau)\; \varphi(\tau)\; d\tau = f(t) \qquad (2\text{-}70)$$

by successive approximations, beginning with the initial approximation $\varphi_o(t) = f(t)$.

This classical approach is included in Appendix III for ready reference.

Also, in Appendix III, Equations (A3-60) and (A3-67), expressions are derived which estimate the truncation error when the infinite expansion is terminated after a finite number of steps. Using the results of Appendix III and letting the exponent n, on $\left[\frac{1}{x(o)}\right]^n$ in equation (2-17) denote the number of the term in the expansion, the truncation of (2-17) at the n^{th} term appears as

$$g_n\,(t) \triangleq f(t) - \frac{1}{x(o)}\int_o^t k(t-\tau)\,f(\tau)\,d\tau + \ldots +$$
$$(-1)^n\left[\frac{1}{x(o)}\right]^n\int_o^t k_n(t-\tau)\,f(\tau)\,d\tau\,. \tag{2-71}$$

If $k(t)$ and $f(t)$ are both bounded on the interval $[o,b]$, i.e.

$$|k(t)| \leq M \tag{2-72}$$

$$|f(t)| \leq m\ , \tag{2-73}$$

equation (A3-67) applies and the truncation error $\{g(t) - g_n(t)\}$ is estimated by

$$|g(t) - g_n(t)| \leq m\left[e^{\left|\frac{1}{x(o)}\right|Mt} - \sum_{j=o}^{n}\frac{\left\{\left|\frac{1}{x(o)}\right|Mt\right\}^j}{j!}\right] \tag{2-74}$$

for $t \varepsilon [o,b]$.

Since the truncation error is the remainder term in the convolution division algorithm of this chapter it is encouraging to observe that as n tends to infinity in (2-74), the remainder term vanishes.

Cases Where the Inversion Simplifies

Because of the relationship presented for convenience in Appendix II between integral equations and differential equations, it is often possible to avoid the infinite expansion of the algorithm, (2-17), by differentiating both sides of (2-1) until the equivalent differential equation for $g(t)$ is recognized. Finding the necessary initial conditions from the successive differentiations, this differential equation is then solved for $g(t)$.

To illustrate, suppose

$$x(t) = \cos(\omega t) \tag{2-75}$$

and

$$y(t) = \sin(\omega t) \tag{2-76}$$

so that (2-1) becomes

$$\sin \omega t = \int_o^t \cos[\omega(t-\tau)]\, g(\tau)\, d\tau \,. \tag{2-77}$$

Differentiating (2-77),

$$\omega \cos \omega t = -\,\omega \int_o^t \sin[\omega(t-\tau)]\, g(\tau)\, d\tau + g(t)\,. \tag{2-78}$$

Differentiating again,

$$- \omega^2 \sin \omega t = -\omega^2 \int_o^t \cos [\omega(t - \tau)] \, g(\tau) \, d\tau + g'(t)$$

which by (2-77) yields the differential equation satisfied by $g(t)$,

$$g'(t) = 0 . \tag{2-79}$$

The differential equation (2-79) has the solution

$$g(t) = \text{constant}$$

and the initial condition term

$$g(o) = \omega ,$$

gotten by setting $t = 0$ in equation (2-78), gives the desired solution

$$g(t) = \omega . \tag{2-80}$$

Evans[14] has studied equations of the same type as (2-48). For convenience his pertinent results are derived in Appendix IV. It is shown there that if the given function $k(t)$ can be recognized as satisfying some differential equation

$$L(k) = 0 \tag{2-81}$$

then it is possible to find the corresponding differential equation

$$\tilde{L}(r) = 0 \tag{2-82}$$

and the initial conditions

$$r^{(n)}(o) = C_n \qquad (2\text{-}83)$$

which $r(t)$ must satisfy.

For the simple elementary functions such as sine, cosine, exponential, etc., these techniques lead to simple differential equations familiar enough to be easily solved. Roughly speaking, this is the same class of functions for which Laplace and Fourier Transforms can be evaluated in closed form. Thus, the standard "text book type" problems where $x(t)$ and $y(t)$ are constructed from familiar elementary functions, are amenable to these shorter, time domain inversions.

These simple problems are quite analogous to the case in ordinary division when the division terminates after a few steps and there is no remainder.

CHAPTER III

A FACTOR ALGORITHM FOR INVERSION

Successive Differentiation

When the convolution integral

$$y(t) = \int_0^t x(t - \tau)g(\tau)d\tau \tag{3-1}$$

is differentiated, using Leibniz's rule*

$$y'(t) = \int_0^t x'(t - \tau)g(\tau)d\tau + x(o)g(t) \tag{3-2}$$

results.

Rearranging (3-2),

$$g(t) = \frac{1}{x(o)}\left\{y'(t) - \int_0^t x'(t - \tau)g(\tau)d\tau\right\} \tag{3-3}$$

and another differentiation yields

$$g'(t) = \frac{1}{x(o)}\left\{y''(t) - \int_0^t x''(t - \tau)g(\tau)d\tau - x'(o)g(t)\right\} . \tag{3-4}$$

Similarly,

---

*See Appendix I, Theorem (A1-II).

$$g''(t) = \frac{1}{x(o)} \left\{ y'''(t) - \int_o^t x'''(t - \tau)g(\tau)d\tau - x''(o)g(t) - x'(o)g'(t) \right\}. \tag{3-5}$$

The pattern is clear and is easily shown by induction to be

$$g^{(n)}(t) = \frac{1}{x(o)} \left\{ y^{(n+1)}(t) - \int_o^t x^{(n+1)}(t - \tau)g(\tau)d\tau - \sum_{j=1}^{n} x^{(j)}(o)g^{(n-j)}(t) \right\} \tag{3-6}$$

where $x^{(m)}(t - \tau)$ denotes the m^{th} derivative of $x(\zeta)$ evaluated at $\zeta = t - \tau$. Equation (3-6) can also be written as

$$g^{(n)}(t) = \frac{1}{x(o)} \left\{ y^{(n+1)}(t) - \int_o^t x^{(n+1)}(t - \tau)g(\tau)d\tau - \sum_{j=o}^{n-1} x^{(n-j)}(o)g^{(j)}(t) \right\}. \tag{3-7}$$

Letting $t = o$ in (3-6), the integral vanishes* and

$$g^{(n)}(o) = \frac{1}{x(o)} \left\{ y^{(n+1)}(o) - \sum_{j=1}^{n} x^{(j)}(o)g^{(n-j)}(o) \right\} \tag{3-8}$$

for $n = 1, 2, \ldots$

with

$$g(o) = \frac{1}{x(o)}\, y'(o). \tag{3-9}$$

Equations (3-8) and (3-9) provide for the determination of the derivatives of $g(t)$ at the point $t = 0$ in terms of the derivatives of the known functions $x(t)$ and $y(t)$ at $t = 0$.

Knowledge of the first $(m + 1)$ derivatives of a function at a point permits the Taylor approximation to the function in a neighborhood of the point.

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*Assuming, of course, that the integrand is proper and contains no impulses.

Thus

$$g(t) = g(o) + g'(o)t + \frac{g^{(2)}(o)t^2}{2!} + \cdots + \frac{g^{(m)}(o)t^m}{m!} + \frac{g^{(m+1)}(\zeta)t^{(m+1)}}{(M+1)!} \tag{3-10}$$

for t close enough to zero that the derivatives are continuous on the interval $[o,t]$ and ζ is some interior point of $[o,t]$.

The last term in (3-10) is called the remainder and is denoted by

$$R_m(t) = \frac{g^{(m+1)}(\zeta)t^{m+1}}{(m+1)!}, \quad o < \zeta < t \,. \tag{3-11}$$

Denoting the first (m+1) terms of (3-10) by

$$P_m(t) = \sum_{j=o}^{m} \frac{g^{(j)}(o)t^j}{j!} \,, \tag{3-12}$$

equation (3-10) is written

$$g(t) = P_m(t) + R_m(t) \tag{3-13}$$

where $P_m(t)$ is the polynomial approximation to $g(t)$ and $R_m(t)$ is the error in this approximation.

Clearly,

$$\lim_{m \to \infty} P_m(t) = g(t) \tag{3-14}$$

if, and only if,

$$\lim_{m \to \infty} R_m(t) = 0 \,. \tag{3-15}$$

A function having continuous derivatives of all orders and satisfying (3-15) for all t in some interval about the origin is said to be analytic at $t = 0$.

For analytic functions, then, (3-8) and (3-9) provide for the approximation of $g(t)$ by a polynomial (3-12) which is valid in a neighborhood of the origin.

The higher the order m of the polynomial $P_m(t)$ the better the approximation

$$g(t) \approx P_m(t). \tag{3-16}$$

For values of t close to the origin only a few terms may be required to approximate $g(t)$.

This method of approximating $g(t)$ near the origin in terms of derivatives of $x(t)$ and $y(t)$ will be used in Chapter IV where practical, real time inversion schemes are developed.

Taylor Series

For analytic functions the Taylor series for $g(t)$ about the origin, (3-17),

$$g(t) = g(o) + g'(o)t + g^{(2)}(o)\,\frac{t^2}{2!} + \cdots + \frac{g^{(n)}(o)t^n}{n!} + \cdots ,$$

is determined by (3-8) and (3-9) from the derivatives of $x(t)$ and $y(t)$. This of course necessitates the existence of derivatives of all orders for the functions $x(t)$ and $y(t)$ and suggests the consideration of the case where x and y are analytic.

Suppose x, y and g are analytic at the origin,

$$x(t) = \sum_{j=o}^{\infty} \frac{x^{(j)}(o)t^j}{j!} \tag{3-18}$$

$$y(t) = \sum_{i=o}^{\infty} \frac{y^{(i)}(o)t^i}{i!} \tag{3-19}$$

$$g(t) = \sum_{n=o}^{\infty} \frac{g^{(n)}(o)t^n}{n!} \quad , \tag{3-20}$$

the series (3-18), (3-19), (3-20) converging absolutely on the interval $o \leq t < b$.

Substituting (3-20) in (3-1),

$$y(t) = \int_o^t x(t - \tau)\left\{\sum_{n=o}^{\infty} \frac{g^{(n)}(o)\tau^n}{n!}\right\} d\tau \tag{3-21}$$

and interchanging the two limit processes,

$$y(t) = \sum_{n=o}^{\infty} \frac{g^{(n)}(o)}{n!} \int_o^t \tau^n x(t - \tau)d\tau \, . \tag{3-22}$$

Now

$$\int_o^t \tau^n x(t - \tau)d\tau = \int_o^t (t - \tau)^n x(\tau)d\tau \tag{3-23}$$

and by Theorem (A1-X), in Appendix I,

$$\frac{1}{n!}\int_0^t (t - \tau)^n x(\tau)d\tau = \underbrace{\int_0^t \cdots \int_0^t}_{(n+1)\text{times}} x(\tau) \underbrace{d\tau \cdots d\tau}_{(n+1)\text{ times}} . \qquad (3\text{-}24)$$

Thus

$$y(t) = g(o) \int_0^t x(\tau)d\tau + g'(o) \int_0^t\int_0^{t_1} x(t_2)dt_2dt_1 + \cdots \qquad (3\text{-}25)$$

$$+ \cdots + g^{(n-1)}(o) \int_0^t\int_0^{t_1}\cdots \int_0^{t_{n-1}} x(t_n)dt_n\cdots dt_2dt_1$$

$$+ \cdots$$

or

$$y(t) = \sum_{n=1}^{\infty} g^{(n-1)}(o) \int_0^t\int_0^{t_1}\cdots \int_0^{t_{n-1}} x(t_n)dt_n\cdots dt_2dt_1 . \qquad (3\text{-}26)$$

Using the notation

$$D^n f(t) \triangleq \frac{d^n}{d\xi^n} f(\xi)\bigg|_{\xi=t} \qquad (3\text{-}27)$$

and

$$D^{-n} f(t) \triangleq \underbrace{\int_0^t \cdots \int_0^t}_{n\text{ times}} f(\varphi) \underbrace{d\varphi\cdots d\varphi}_{n\text{ times}} , \qquad (3\text{-}28)$$

equation (3-26) may be written more compactly as

$$y(t) = \sum_{n=o}^{\infty} D^n g(o) D^{-(n+1)} x(t) . \qquad (3\text{-}29)$$

So far, only the analyticity of $g(t)$, (3-21), and the integrability of $x(t)$, (3-22), are necessary for the validity of (3-25) or (3-29).

If the analyticity of $x(t)$ is now used and (3-18) is substituted in (3-23), then

$$\int_o^t (t - \tau)^n x(\tau) d\tau = \int_o^t (t - \tau)^n \left\{ \sum_{j=o}^{\infty} \frac{x^{(j)}(o)\tau^j}{j!} \right\} d\tau \quad . \tag{3-30}$$

Interchanging operations yields

$$\int_o^t (t - \tau)^n x(\tau) d\tau = \sum_{j=o}^{\infty} \frac{x^{(j)}(o)}{j!} \int_o^t (t - \tau)^j \tau^n d\tau \quad . \tag{3-31}$$

But, by (3-24),

$$\frac{1}{j!} \int_o^t (t - \tau)^j \tau^n d\tau = D^{-(j+1)} t^n \tag{3-32}$$

and integrating t^n from o to t the indicated $(j + 1)$ times it follows that

$$\frac{1}{j!} \int_o^t (t - \tau)^j \tau^n d\tau = \frac{n!}{(n + j + 1)!} t^{(n+j+1)} . \tag{3-33}$$

Thus (3-22) becomes

$$y(t) = \sum_{n=o}^{\infty} \frac{g^{(n)}(o)}{n!} \left\{ \sum_{j=o}^{\infty} x^{(j)}(o) \frac{n! t^{(n+j+1)}}{(n + j + 1)!} \right\} \tag{3-34}$$

and the assumed absolute convergence permits the rearrangement

$$y(t) = \sum_{n=o}^{\infty} \sum_{j=o}^{\infty} \frac{g^{(n)}(o)x^{(j)}(o)}{(n + j + 1)!} t^{(n+j+1)} \quad . \tag{3-35}$$

Expanding (3-35) in ascending powers of t gives

$$\begin{aligned} y(t) = {} & \frac{g(o)x(o)}{1!} t + \left\{ \frac{g(o)x'(o)}{2!} + \frac{g'(o)x(o)}{2!} \right\} t^2 \\ & + \left\{ \frac{g(o)x^{(2)}(o)}{3!} + \frac{g'(o)x'(o)}{3!} + \frac{g^{(2)}(o)x(o)}{3!} \right\} t^3 \\ & + \cdots + \frac{1}{(k+1)!} \left\{ \sum_{j=o}^{k} g^{(j)}(o)x^{(k-j)}(o) \right\} t^{(k+1)} \\ & + \cdots . \end{aligned} \tag{3-36}$$

Writing (3-36) as

$$y(t) = \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \sum_{j=o}^{(k-1)} g^{(j)}(o)x^{(k-1-j)}(o) \right\} t^k \tag{3-37}$$

and comparing with (3-19), it follows by the uniqueness of the power series expansion that

$$y^{(k+1)}(o) = \sum_{j=o}^{k} g^{(j)}(o)x^{(k-j)}(o) \tag{3-38}$$

for $k = 0, 1, 2, \ldots$.

It is observed that (3-38) is a convolution summation of the orders of differentiation of g(t) and x(t) required to produce a given order of differentiation of $y'(t)$ at the origin.

When (3-38) is solved for the $\left\{g^{(i)}(o)\right\}$ in terms of the $\left\{x^{(i)}(o)\right\}$ and $\left\{y^{(i)}(o)\right\}$,

$$g(o) = \frac{1}{x(o)}\, y'(o) \tag{3-39}$$

and

$$g^{(n)}(o) = \frac{1}{x(o)}\left\{y^{(n+1)}(o) - \sum_{j=1}^{n} x^{(j)}(o)g^{(n-j)}(o)\right\} \tag{3-40}$$

for $n = 1, 2, \ldots$,

the same recursive solution as equations (3-8) and (3-9).

The Factor Algorithm

The Taylor series expansion for an analytic function $f(t)$

$$f(t) = f(o) + f'(o)t + f^{(2)}(o)\,\frac{t^2}{2!} + \cdots + f^{(n)}(o)\,\frac{t^n}{n!} + \cdots \tag{3-41}$$

may be thought of as an expansion of the function into its component unit singularity functions* the unit step, ramp, parabola, cubic and so on.

Using the conventional notation for the unit singularity functions

$$\mu_o(t) = \delta(t) \qquad \text{(the unit impulse)} \tag{3-42}$$

$$\mu_{-1}(t) = \int_o^t \delta(\tau)d\tau \qquad \text{(the unit step)} \tag{3-43}$$

$$\mu_{-2}(t) = \int_o^t \mu_{-1}(\tau)d\tau \qquad \text{(the unit ramp)} \tag{3-44}$$

---

*So called because they may be generated by successive differentiations or integrations of the unit impulse.

and

$$\mu_{-n}(t) = \int_0^t \mu_{-(n-1)}(\tau)d\tau \qquad (3\text{-}45)$$

it is seen that, for $t \geq 0$,

$$\mu_{-1}(t) = 1 \qquad \text{(step)} \qquad (3\text{-}46)$$

$$\mu_{-2}(t) = t \qquad \text{(ramp)} \qquad (3\text{-}47)$$

$$\mu_{-3}(t) = \frac{1}{2}t^2 \qquad \text{(parabola)} \qquad (3\text{-}48)$$

$$\mu_{-4}(t) = \frac{1}{3!}\, t^3 \qquad \text{(cubic)} \qquad (3\text{-}49)$$

$$\mu_{-n}(t) = \frac{1}{(n-1)!}\, t^{(n-1)}\, . \qquad (3\text{-}50)$$

From the definition (3-45) and equation (3-33) it follows that

$$\mu_{-(n+1)}(t) = \int_0^t \mu_{-1}(t - \tau)\mu_{-n}(\tau)d\tau \qquad (3\text{-}51)$$

and that

$$\mu_{-(n+j)}(t) = \int_0^t \mu_{-j}(t - \tau)\mu_{-n}(\tau)d\tau \quad . \qquad (3\text{-}52)$$

In terms of the unit singularity functions the Taylor series (3-41) may then be written as

$$f(t) = f(o)\mu_{-1}(t) + f'(o)\mu_{-2}(t) + f^{(2)}(o)\mu_{-3}(t) + \cdots \qquad (3\text{-}53)$$

$$+ f^{(n)}(o)\, \mu_{-(n+1)}(t) + \cdots$$

or

$$f(t) = \sum_{n=o}^{\infty} f^{(n)}(o)\mu_{-(n+1)}(t) \quad . \tag{3-54}$$

With this interpretation equation (3-38) is thought of as showing how the unit singularity components of $x(t)$ and $g(t)$ combine to produce a given unit singularity component of $y(t)$.

For example, there is no unit step in $y(t)$ since the lowest order singularity components in $x(t)$ and $g(t)$ are the unit steps $x(o)\mu_{-1}(t)$ and $g(o)\mu_{-1}(t)$, respectively, and they convolve by (3-51) to give the unit ramp

$$x(o)g(o)\mu_{-2}(t) \quad .$$

Thus,

$$y(o) = 0 \tag{3-55}$$

and

$$y'(o) = x(o)g(o) \quad . \tag{3-56}$$

Similarly, the unit parabola, $y^{(2)}(o)\mu_{-3}(t)$, in $y(t)$ is due to two sources:

the step in x convolving with the ramp in g,

the ramp in x convolving with the step in g.

Thus

$$y^{(2)}(o) = x(o)g'(o) + x'(o)g(o) \quad . \tag{3-57}$$

With $g(o)$ determined from (3-66) as

$$g(o) = \frac{1}{x(o)} y'(o) \quad , \tag{3-58}$$

equation (3-57) is then solvable for g'(o) as

$$g'(o) = \frac{1}{x(o)} \left\{ y^{(2)}(o) - \frac{x'(o)y'(o)}{x(o)} \right\} \tag{3-59}$$

The determination of equations (3-55), (3-56) and (3-57) and the solutions (3-58) and (3-59) illustrates how the "factor" algorithm of this section is carried out.

The process is identical to the algebraic problem where a polynomial y(t) is recognized as the product of two other polynomials x(t) and g(t) — the "factors" of y(t)— and the coefficients of y(t) are related to those of x(t) and g(t).

This comparison will be strengthened by the discussion in Chapter V where an infinite series is expressed as the product of two factors each of which is an infinite series and the solution is carried out by long division.

CHAPTER IV

REAL TIME INVERSION

It was pointed out in Chapter I that the customary inversion of convolution by Laplace Transforms was not a "real time" technique because $x(t)$ and $y(t)$ had to be known for all time, $o \leq t < \infty$, in order to compute $g(t)$ by equation (1-18).

In this Chapter methods will be derived which permit the computation of $g(t)$ from the knowledge of the waveforms $x(t)$ and $y(t)$ on the interval $[o,t]$ only. Since $g(t)$ is thus determined at time t by the previous t seconds of $x(t)$ and $y(t)$, these methods are regarded as being performed in "real time" in the sense that a running estimate of $g(t)$ is made from the normal operating records rather than waiting until the entire waveforms for $x(t)$ and $y(t)$ are known before beginning the computations.

In fact, it will now be shown that convolution may be inverted as rapidly and as easily as it may be performed.

The Rectangular Prototype

In Chapter I, Figure 2 illustrates graphically how the convolution integral

$$y(t) = \int_0^t x(t-\tau)g(\tau)d\tau \qquad (4\text{-}1)$$

represents $y(t)$ as the area under the product of the two waveforms $x(t-\tau)$ and $g(\tau)$.

If the waveforms $x(t)$ and $g(t)$ are approximated by sufficiently simple functions the integration of equation (4-1) simplifies.

For example, if x(t) and g(t) are each approximated by the staircase or rectangular waveforms denoted by $x^*(t)$ and $g^*(t)$, Figure 2d will appear as Figure 9.

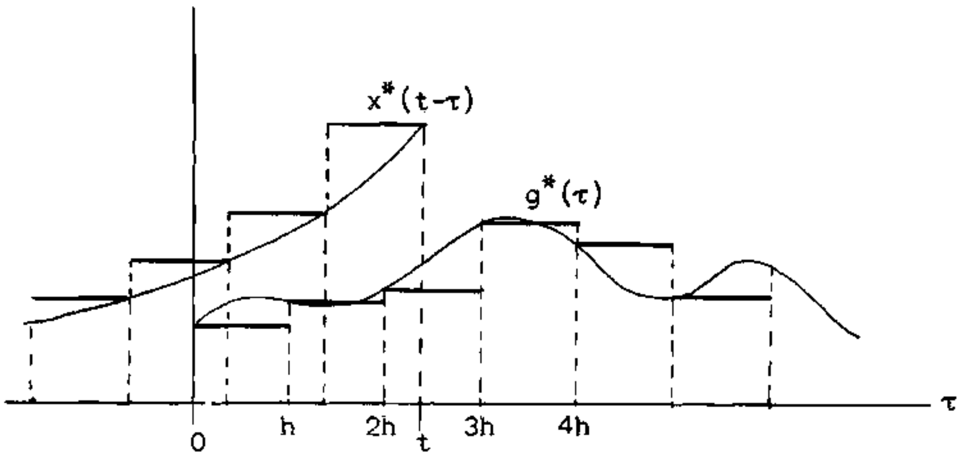

Figure 9. Staircase Approximations Used in Convolution.

If y(t) is computed only at those values of t=o, h, 2h, ..., nh, ..., where h is the sampling interval of the rectangular pulse train comprising the staircase approximation, the integration reduces to a finite sum and the following system of linear equations results.

$$y(o) = 0$$

$$y(h) \doteq h\, x(o)g(o)$$

$$y(2h) \doteq h\, [x(o)g(h) + x(h)g(o)]$$

$$y(3h) \doteq h[x(o)g(2h) + x(h)g(h) + x(2h)g(o)]$$

$$\vdots$$

(continued)

$$y([n+1]h) \doteq h \sum_{j=o}^{n} x(jh)g([n - j]h) .$$

Clearly, $y(o) = 0$ assumes that neither $x(t)$ nor $g(t)$ contains an impulse at the origin.

Introducing the compact notation

$$f_n \triangleq f(nh) \tag{4-2}$$

$$\text{for } n = 0,1,2,\ldots ,$$

this linear system may be written

$$\begin{aligned}
y_1 &\doteq h\,[x_o g_o] \\
y_2 &\doteq h\,[x_o g_1 + x_1 g_o] \\
y_3 &\doteq h\,[x_o g_2 + x_1 g_1 + x_2 g_o] \\
&\ \ \vdots \\
y_{n+1} &\doteq h \sum_{j=o}^{n} x_j g_{n-j}
\end{aligned} \tag{4-3}$$

$$\text{for } n=0,\ 1,\ 2,\ \ldots\ .$$

Thus, subject to the accuracy of the staircase approximations to $x(t)$ and $g(t)$, the values of $y(t)$ at the sampling instants,

$$t_n \triangleq nh \tag{4-4}$$

$$\text{for } n = 0, 1, 2, \ldots ,$$

is given by (4-3). Since the convolution of two rectangular pulses is a triangular pulse, y(t) is approximated between the sampling instants by

$$y(t) \doteq \frac{y_{n+1} - y_n}{h} (t - t_n) + y_n \qquad (4\text{-}5)$$

$$\text{for } t_n \leq t \leq t_{n+1} .$$

Equations (4-3) and (4-5), then, provide the piecewise linear approximation to y(t), denoted by $y^{**}(t)$, from the rectangular approximations, $x^*(t)$ and $g^*(t)$, of the waveforms x(t) and g(t).

It is apparent, however, that the system (4-3) is just as useful in solving equation (4-1) for $g^*(t)$ in terms of the known waveforms of x(t) and y(t):

The first equation in (4-3) may be solved for g_o in terms of x_o and y. The second equation in (4-3) uses this value of g_o to solve for g_1 in terms of x_o, x_1, y_2 and g_o. The third equation yields g_2, and so on.

It is also apparent that this solution is performed in real time, using as it does only those values of x(t) and y(t) in the interval $[o,(n+1)h]$ to compute g(nh).

The approximate real time inversion of equation (4-1) is thus accomplished as simply as convolution itself by the staircase approximation of the waveforms involved.

Solving (4-3) for the g_n as indicated yields #

$$g_o \doteq \frac{1}{x_o} [\frac{y_1}{h}] \qquad \text{(continued)}$$

#It is obviously assumed that $x_o \neq o$ and it is equally obvious from (4-3) that this is no restriction on the method.

$$g_1 \doteq \frac{1}{x_o} [\frac{y_2}{h} - x_1 g_o]$$

$$g_2 \doteq \frac{1}{x_o} [\frac{y_3}{h} - x_1 g_1 - x_2 g_o] \tag{4-6}$$

$$\vdots$$

$$g_n \doteq \frac{1}{x_o} [\frac{y_{n+1}}{h} - \sum_{j=o}^{n-1} g_j x_{n-j}]$$

for $n = 1, 2, \ldots\ldots$.

<u>Other Approximations to Waveforms</u>

The use of more involved approximations to the waveforms leads to more involved linear alegraic systems than (4-3) to be solved for the approximation to $g(t)$. Two common useful approximating schemes will now be considered.

If $x(t)$ is approximated by the staircase $x^*(t)$ and $g(t)$ by the piecewise linear waveform $g^{**}(t)$, as illustrated in Figure 10 for the computation of $y(2h)$, the convolution integral (4-1) yields the linear system (4-7).

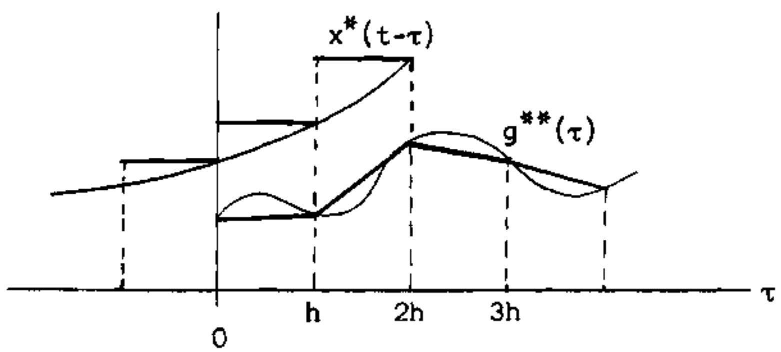

Figure 10. Convolving x* and g** for y(2h).

$$y_1 \doteq \frac{1}{2} h \, [x_o(g_1 + g_o)]$$

$$y_2 \doteq \frac{1}{2} h \, [x_o(g_2 + g_1) + x_1(g_1 + g_o)]$$

$$y_3 \doteq \frac{1}{2} h \, [x_o(g_3 + g_2) + x_1(g_2 + g_1) + x_2(g_1 + g_o)]$$

$$\vdots \tag{4-7}$$

$$y_{n+1} \doteq \frac{1}{2} h \sum_{j=o}^{n} (g_{j+1} + g_j) x_{n-j}$$

for $n = 0, 1, 2, \ldots$.

The system (4-7) may be solved for the $\{g_n\}$ only if there is some means for determining g_o to start the solution. This is accomplished quite simply by recognizing that for ε sufficiently small,

$$y(\varepsilon) \doteq \varepsilon \; x(o)g(o) \tag{4-8}$$

or

$$g(o) \doteq \frac{1}{x(o)} \frac{y(\varepsilon)}{\varepsilon} \quad \text{for } \varepsilon \text{ very small.} \tag{4-9}$$

This expression is clearly equivalent to requiring knowledge of $y'(t)$ at $t=o$ and should be compared with equation (3-9).

The solution for $g^{**}(t)$ is thus

$$g_o \doteq \frac{1}{x(o)} \frac{y(\varepsilon)}{\varepsilon} \quad \text{for } \varepsilon << h$$

$$g_1 \doteq \frac{1}{x_o} \left[\frac{2}{h} y_1 \right] - g_o \qquad \text{(continued)} \tag{4-10}$$

$$g_2 \doteq \frac{1}{x_o} \left[\frac{2}{h} y_2 - x_1(g_1 + g_o) \right] - g_1$$

$$g_3 \doteq \frac{1}{x_o} \left[\frac{2}{h} y_3 - x_1(g_2 + g_1) - x_2(g_1 + g_o) \right] - g_2$$

$$\vdots$$

$$g_n \doteq \frac{1}{x_o} \left[\frac{2}{h} y_n - \sum_{j=1}^{n-1} (g_j + g_{j-1}) x_{n-j} \right] - g_{n-1}$$

for $n = 2, 3, \ldots\ldots$.

If both x(t) and g(t) are approximated by the piecewise linear waveforms $x^{**}(t)$ and $g^{**}(t)$, as shown in Figure 11 for the computation of y(2h), the convolution integral yields the linear system (4-11).

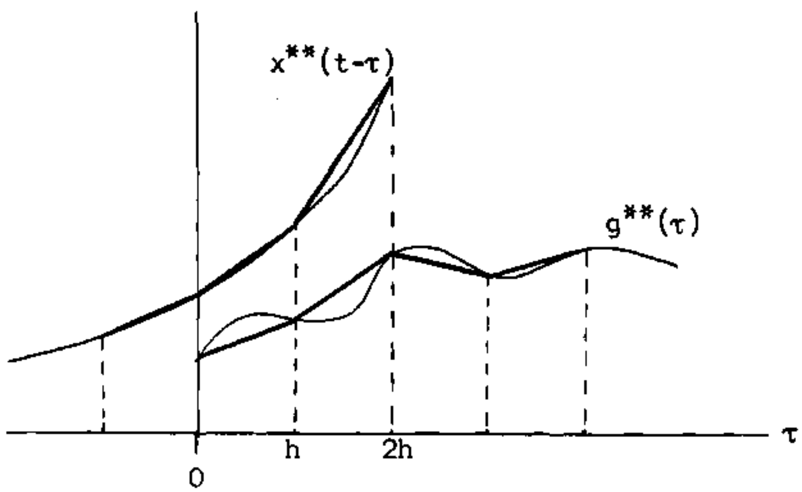

Figure 11. Convolving x** and g** for y(2h).

$$
\begin{aligned}
y_1 &\doteq \frac{h}{6}\left\{g_o[2x_1 + x_o] + g_1[x_1 + 2x_o]\right\} \\
y_2 &\doteq \frac{h}{6}\left\{g_o[2x_2 + x_1] + g_1[x_o + 4x_1 + x_2] + g_2[x_1 + 2x_o]\right\} \\
y_3 &\doteq \frac{h}{6}\left\{g_o[2x_3 + x_2] + g_1[x_1 + 4x_2 + x_3] + \right. \\
&\qquad \left. + g_2[x_o + 4x_1 + x_2] + g_3[x_1 + 2x_o]\right\} \\
y_4 &\doteq \frac{h}{6}\left\{g_o[2x_4 + x_3] + g_1[x_2 + 4x_3 + x_4] + \right. \\
&\qquad + g_2[x_1 + 4x_2 + x_3] + \\
&\qquad \left. + g_3[x_o + 4x_1 + x_2] + g_4[x_1 + 2x_o]\right\} \\
&\cdot \\
&\cdot \\
&\cdot \\
y_n &\doteq \frac{h}{6}\left\{g_o[2x_n + x_{n-1}] + \sum_{j=1}^{n-1} g_j[x_{n+1-j} + 4x_{n-j} + x_{n-1-j}] + \right. \\
&\qquad \left. + g_n[x_1 + 2x_o]\right\}
\end{aligned}
\tag{4-11}
$$

for $n = 1, 2, \ldots$.

Once again the solution of system (4-11) must be started by an auxiliary computation of g_o as in (4-9). This is equivalent to saying that the expression for y_n in (4-11) is a linear difference equation whose solution for g_n requires the specification of the initial condition term g_o.

The solution of (4-11) for g** is

$$g_o \doteq \frac{1}{x(o)} \; \frac{y(\varepsilon)}{\varepsilon} \quad \text{for } \varepsilon << h$$

$$g_1 \doteq \frac{\frac{6}{h} y_1 - g_o(2x_1 + x_o)}{(x_1 + 2x_o)}$$

$$g_2 \doteq \frac{\frac{6}{h} y_2 - [g_o(2x_2 + x_1) + g_1(x_o + 4x_1 + x_2)]}{(x_1 + 2x_o)} \tag{4-12}$$

$$g_3 \doteq \frac{\frac{6}{h} y_3 - [g_o(2x_3 + x_2) + g_1(x_1 + 4x_2 + x_3) + g_2(x_o + 4x_1 + x_2)}{(x_1 + 2x_o)}$$

.

.

.

$$g_n \doteq \frac{\frac{6}{h} y_n - \left[g_o(2x_n + x_{n-1}) + \sum_{j=1}^{n-1} g_j[x_{n+1-j} + 4x_{n-j} + x_{n-1-j}] \right]}{(x_1 + 2x_o)}$$

for n = 1, 2, ...

To facilitate computation, the symbols

$$D \triangleq (x_1 + 2x_o) \tag{4-13}$$

$$R_n \triangleq (2x_n + x_{n-1}) \tag{4-14}$$

$$S_n \triangleq (x_n + 4x_{n+1} + x_{n+2}) \tag{4-15}$$

permit the solution system (4-12) to be written

$$g_o \doteq \frac{1}{x(o)} \frac{y(\varepsilon)}{\varepsilon} \qquad \text{for } \varepsilon << h$$

$$g_1 \doteq \frac{\frac{6}{h} y_1 - [g_o R_1]}{D}$$

$$g_2 \doteq \frac{\frac{6}{h} y_2 - [g_o R_2 + g_1 S_o]}{D}$$

$$g_3 \doteq \frac{\frac{6}{h} y_3 - [g_o R_3 + g_1 S_1 + g_2 S_o]}{D} \qquad (4\text{-}16)$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$g_n \doteq \frac{\frac{6}{h} y_n - \left[g_o R_n + \sum_{j=1}^{n-1} g_j S_{n-1-j} \right]}{D}$$

for $n = 2, 3, \ldots$.

For continuous waveforms the piecewise linear solution (4-16) will of course be a more accurate approximation than either (4-6) or (4-10).

Since the piecewise linear approximations of waveforms are so widely used in circuit theory and signal analysis(8), usually in the approximate calculation of Laplace or Fourier transforms, the details of the approximate convolution, (4-11) and the approximate inversion (4-12) will now be discussed.

The integration of the product of two piecewise linear waveforms yielding system (4-11) resulted from the repeated use of the basic formulas (4-19) and (4-20) whose derivation is clear from Figure 12.

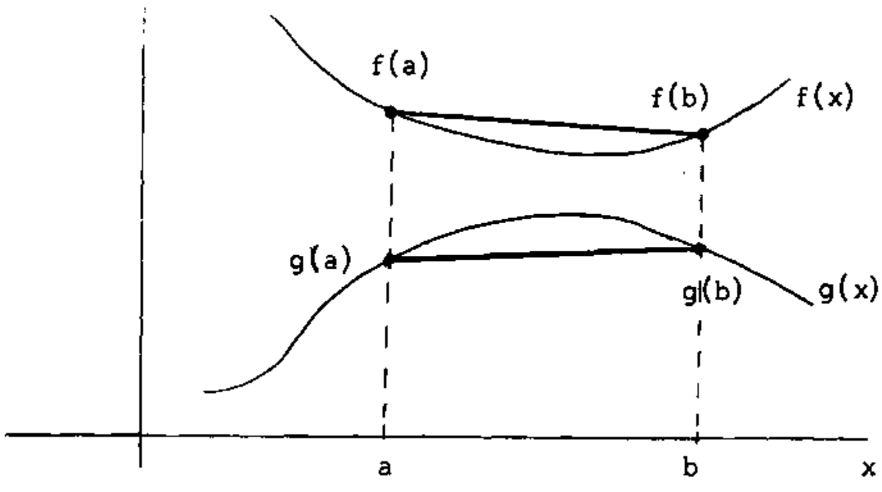

Figure 12. Product Integration and Linear Approximation.

Since

$$f(x) \doteq \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \tag{4-17}$$

and

$$g(x) \doteq \frac{g(b) - g(a)}{b - a}(x - a) + g(a) , \tag{4-18}$$

it follows that

$$\int_a^b f(x)g(x)dx \doteq \frac{(b - a)}{3}\left[g(a)f(a) + \frac{g(a)f(b) + g(b)f(a)}{2} + g(b)f(b)\right] \tag{4-19}$$

or, equivalently,

$$\int_a^b f(x)g(x)dx \doteq \frac{(b-a)}{6}\left[g(a)[2f(a) + f(b)] + g(b)[f(a) + 2f(b)]\right]. \quad (4\text{-}20)$$

When the interval of integration, $[a,b]$, is uniformly partitioned into n subintervals, each of length

$$h = \frac{b - a}{n} \quad (4\text{-}21)$$

and the notation (4-2) is used, the repeated application of (4-20) yields

$$\int_{x_o}^{x_n} f(x)g(x)dx \doteq \frac{h}{6}\left\{g_o[2f_o + f_1] + \sum_{j=1}^{n-1} g_j[f_{j-1} + 4f_j + f_{j+1}] + g_n[f_{n-1} + 2f_n]\right\} . \quad (4\text{-}22)$$

It should be mentioned that equation (4-19) bears a resemblance to the Simpson rule formula[15]

$$\int_a^b f(x)g(x)dx \doteq \frac{b - a}{3}\left[g(a)f(a) + 4g(\tfrac{a + b}{2})f(\tfrac{a + b}{2})\; g(b)f(b)\right] . \quad (4\text{-}23)$$

This is not surprising since both formulas involve a parabolic approximation to the integrand $f(x)g(x)$.

Notice, however, that the Simpson formula requires the knowledge of the functions $f(x)$ and $g(x)$ at an additional sample point, $x = \frac{a + b}{2}$, while (4-19) requires only the two sample points $x = a$ and $x = b$.

Approximate Quadratures

Instead of approximating each of the waveforms $x(t)$ and $g(t)$ and then multiplying the approximations and integrating as in the methods just discussed, the standard approximate quadrature formulas[16] approximate the product function $x(t - \tau)g(\tau)$ directly. When these approximate quadratures are used in the convolution integral, (4-1), there result linear approximating systems whose form and solution resemble those of (4-3), (4-7) and (4-11).

The simplest approximate quadrature is the rectangular rule and when applied to (4-1) it yields

$$
\begin{aligned}
y_o &= 0 \\
y_1 &\doteq h[x_1 g_o] \\
y_2 &\doteq h[x_2 g_o + x_1 g_1] \\
y_3 &\doteq h[x_3 g_o + x_2 g_1 + x_1 g_2] \\
&\vdots \\
y_{n+1} &\doteq h \sum_{j=o}^{n} x_{n+1-j} g_j, \qquad n = 1, 2, \ldots ,
\end{aligned}
\tag{4-24}
$$

as is readily visualized from Figure 13.

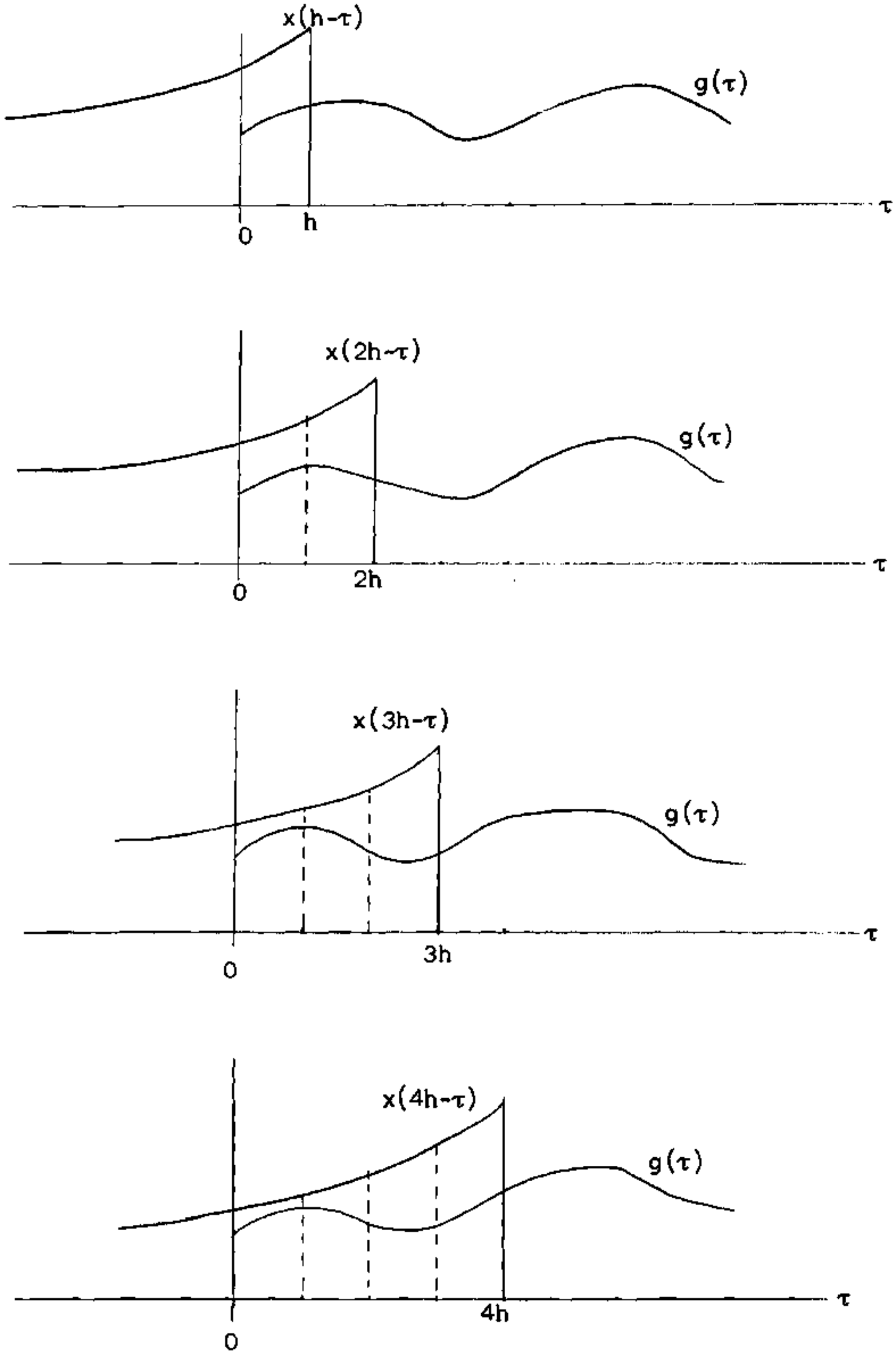

Figure 13. An Aid in Approximate Integration.

This system is of course very similar to that of (4-3) and its solution is

$$g_o \doteq \frac{1}{x_1}\left[\frac{y_1}{h}\right]$$

$$g_1 \doteq \frac{1}{x_1}\left[\frac{y_2}{h} - x_2 g_o\right]$$

$$g_2 \doteq \frac{1}{x_1}\left[\frac{y_3}{h} - x_3 g_o - x_2 g_1\right] \tag{4-25}$$

$$\vdots$$

$$g_n \doteq \frac{1}{x_1}\left[\frac{y_{n+1}}{h} - \sum_{j=o}^{n-1} x_{n+1-j} g_j\right]$$

for $n = 1, 2, \ldots$.

For continuous functions, a more accurate quadrature is provided by the trapezoidal rule which, approximating the integral by a sequence of trapezoidal pulses, gives

$$\int_{x_o}^{x_n} f(x)dx \doteq h\left[\frac{1}{2} f(x_o) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right] \tag{4-26}$$

for the integral of $f(x)$ over the n uniformly partitioned subintervals of $[x_o, x_n]$.

With the aid of Figure 13, formula (4-26) applied to (4-1) yields

$$y_o = 0$$

$$y_1 \doteq h[\tfrac{1}{2}\, x_o g_1 + \tfrac{1}{2}\, x_1 g_o]$$

$$y_2 \doteq h[\tfrac{1}{2}\, x_o g_2 + x_1 g_1 + \tfrac{1}{2}\, x_2 g_o] \tag{4-27}$$

$$y_3 \doteq h[\tfrac{1}{2}\, x_o g_3 + x_1 g_2 + x_2 g_1 + \tfrac{1}{2}\, x_3 g_o]$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$y_n \doteq h \sum_{j=o}^{n} g_j x_{n-j} - \frac{h}{2}\,(x_o g_n + x_n g_o)$$

for $n = 1, 2, \ldots$.

Estimating the initial condition term g_o as before, the solution of system (4-27) is

$$g_o \doteq \frac{1}{x_o}\,\frac{y(\varepsilon)}{\varepsilon} \qquad \text{for } \varepsilon << h$$

$$g_1 \doteq \frac{2}{x_o}\,[\frac{y_1}{h} - \tfrac{1}{2}\, x_1 g_o]$$

$$g_2 \doteq \frac{2}{x_o}\,[\frac{y_2}{h} - x_1 g_1 - \tfrac{1}{2}\, x_2 g_o] \tag{4-28}$$

$$g_3 \doteq \frac{2}{x_o}\,[\frac{y_3}{h} - x_1 g_2 - x_2 g_1 - \tfrac{1}{2}\, x_3 g_o]$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$g_n \doteq \frac{2}{x_o}\left[\frac{y_n}{h} - \frac{1}{2}\, x_n g_o - \sum_{j=1}^{n-1} g_j x_{n-j}\right]$$

for n = 2, 3, ...

Probably the most widely used quadrature in numerical analysis is <u>Simpson's rule</u>. As has already been mentioned in connection with (4-23), this rule involves fitting a sequence of parabolic pulses through adjacent triplets of sample points and requires an even number of sampling intervals.

Simpson's formula for the integral of a function f(x) over the n equally spaced intervals from x_o to x_n is

$$\int_{x_o}^{x_n} f(x)dx \doteq \frac{h}{3}\Big\{f(x_o) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big\} \quad (4\text{-}29)$$

for n even.

With the aid of Figure 13, the calculation of (4-1) using Simpson's rule yields

$$\begin{aligned} y_o &= 0 \\ y_2 &\doteq \frac{h}{3}\,[g_o x_2 + 4g_1 x_1 + g_2 x_o] \ , \\ y_4 &\doteq \frac{h}{3}\,[g_o x_4 + 4g_1 x_3 + 2g_2 x_2 + 4g_3 x_1 + g_4 x_o] \ , \\ y_6 &\doteq \frac{h}{3}\,[g_o x_6 + 4g_1 x_5 + 2g_2 x_4 + 4g_3 x_3 + 2g_4 x_2 + 4g_5 x_1 + g_6 x_o] \ . \end{aligned} \quad (4\text{-}30)$$

This linear algebraic system, however, is not solvable for the $\{g_n\}$ since each equation in (4-30) has two more unknowns than its predecessor.

In the trapezoidal rule system, (4-27), an unsolvable system was made solvable by the auxiliary computation of an initial condition term, g_o, from the relation (4-8).

To make (4-30) solvable, it will first be necessary to generate some additional equations, namely those for $\{y_1, y_3, y_5, \dots\}$. This will be accomplished by combining the trapezoidal rule and the Simpson rule with the former being used to approximate the area over the interval $[o,h]$ and the latter for the remaining area.

For example, with the aid of Figure 13, formulas (4-26) and (4-29) yield

$$y_1 \doteq \frac{h}{2}[g_ox_1 + g_1x_o] \quad ,$$

$$y_3 \doteq \frac{h}{2}[g_ox_3 + g_1x_2] + \frac{h}{3}[g_1x_2 + 4g_2x_1 + g_3x_o] \quad , \qquad (4\text{-}31)$$

$$y_5 \doteq \frac{h}{2}[g_ox_5 + g_1x_4] + \frac{h}{3}[g_1x_4 + 4g_2x_3 + 2g_3x_2 + 4g_4x_1 + g_5x_o]$$

for n odd.

The hybrid system (4-31) combined with (4-30) and the initial condition (4-9) for g_o is the now solvable system

$$y_o = 0$$

$$y_1 \doteq \frac{h}{2}[g_ox_1 + g_1x_o]$$

$$y_2 \doteq \frac{h}{3}[g_ox_2 + 4g_1x_1 + g_2x_o]$$

$$y_3 \doteq \frac{h}{2}[g_ox_3 + g_1x_2] + \frac{h}{3}[g_1x_2 + 4g_2x_1 + g_3x_o] \qquad \text{(continued)}$$

$$y_4 \doteq \frac{h}{3}\left[g_o x_4 + 4g_1 x_3 + 2g_2 x_2 + 4g_3 x_1 + g_4 x_o\right]$$

$$y_5 \doteq \frac{h}{2}\left[g_o x_5 + g_1 x_4\right] + \frac{h}{3}\left[g_1 x_4 + 4g_2 x_3 + 2g_3 x_2 + 4g_4 x_1 + g_5 x_o\right]$$

$$y_6 \doteq \frac{h}{3}\left[g_o x_6 + 4g_1 x_5 + 2g_2 x_4 + 4g_3 x_3 + 2g_4 x_2 + 4g_5 x_1 + g_6 x_o\right]$$

$$\vdots \qquad\qquad (4\text{-}32)$$

$$y_n \doteq \frac{h}{2}\left[g_o x_n + g_1 x_{n-1}\right] + \frac{h}{3}\left[(g_1 x_{n-1} + g_n x_o) + 2\sum_{j=1}^{n-2} g_{j+1} x_{n-1-j} + 2\sum_{j=1}^{\frac{1}{2}(n-1)} g_{2j} x_{n-2j}\right]$$

for n odd,

$$y_n \doteq \frac{h}{3}\left[(g_o x_n + g_n x_o) + 2\sum_{j=1}^{n-1} g_j x_{n-j} + 2\sum_{j=1}^{\frac{1}{2}n} g_{2j-1} x_{n+1-2j}\right]$$

for n even.

The solution of this hybrid system is

$$g_o \doteq \frac{1}{x(o)} \frac{y(\varepsilon)}{\varepsilon} \qquad \text{for } \varepsilon << h$$

$$g_1 \doteq \frac{1}{x_o} [\frac{2}{h} y_1 - g_o x_1]$$

$$g_2 \doteq \frac{1}{x_o} [\frac{3}{h} y_2 - g_o x_2 - 4g_1 x_1]$$

$$g_3 \doteq \frac{1}{x_o} \left[\frac{3y_3}{h} - \frac{3}{2} (g_o x_3 + g_1 x_2) - (g_1 x_2 + 4g_2 x_1) \right]$$

$$g_4 \doteq \frac{1}{x_o} \left[\frac{3y_4}{h} - (g_o x_4 + 4g_1 x_3 + 2g_2 x_2 + 4g_3 x_1) \right] \tag{4-33}$$

$$g_5 \doteq \frac{1}{x_o} \left[\frac{3y_5}{h} - \frac{3}{2} (g_o x_5 + g_1 x_4) - (g_1 x_4 + 4g_2 x_3 + 2g_3 x_2 + 4g_4 x_1) \right]$$

.

.

.

$$g_n \doteq \frac{1}{x_o} \left[\frac{3y_n}{h} - \left(g_o x_n + 2 \sum_{j=1}^{n-1} g_j x_{n-j} + 2 \sum_{j=1}^{\frac{1}{2}n} g_{2j-1} x_{n+1-2j} \right) \right]$$

for n even,

and

$$g_n \doteq \frac{1}{x_o} \left[\frac{3y_n}{h} - \frac{3}{2} (g_o x_n + g_1 x_{n-1}) - g_1 x_{n-1} - 2 \sum_{j=1}^{n-2} g_{j+1} x_{n-1-j} - 2 \sum_{j=1}^{\frac{1}{2}(n-1)} g_{2j} x_{n-2j} \right]$$

for n odd.

This solution, (4-33), may be regarded as a refinement of the trapezoidal rule for smooth functions and hence is useful even though it does not attain the full power of Simpson's parabolic approximation over the whole interval of integration.

Error Bounds

All of the approximations of this chapter have resulting in approximating the convolution integral (4-1) by an expression of the form

$$y(t) \doteq \sum_{j=o}^{n} A_j x(t - jh) g(jh) \tag{4-34}$$

$$\text{for } o \leq t \leq nh \quad ,$$

where the $\{A_j\}$ are constants determined by the approximation scheme employed.*

Denoting the error in the approximation of $y(t)$ by $\rho(t)$, equation (4-1) may be written as the equality

$$y(t) + \rho(t) = \sum_{j=o}^{n} A_j x(t - jh) g(jh) \tag{4-35}$$

$$\text{for } o \leq t \leq nh.$$

Computing $y(t)$ at the sampling instants yields

$$y(nh) + \rho(nh) = \sum_{j=o}^{n} A_j x(nh - jh) g(jh) \tag{4-36}$$

*This method of error analysis is discussed in Kantorovich and Krylov, Ref. (16).

or, in shorter notation, the system

$$y_n + \rho_n = \sum_{j=o}^{n} A_j x_{n-j} g_j \; . \tag{4-37}$$

Now the solution of the approximate system,

$$y_n \doteq \sum_{j=o}^{n} A_j x_{n-j} g_j \tag{4-38}$$

can be expressed by Cramer's rule as

$$\tilde{g}_j = \frac{1}{\Delta} \sum_{k=o}^{n} \Delta_{kj} \, y_k \quad , \tag{4-39}$$

$$\text{for } j = 0, 1, \ldots, n \quad ,$$

where Δ is the determinant of the matrix of system (4-38) and Δ_{kj} is the minor of the element in the k^{th} row and j^{th} column of the system matrix.

Now the exact solution is gotten in precisely the same manner from the exact system (4-37) and is

$$g_j = \frac{1}{\Delta} \sum_{k=o}^{n} \Delta_{kj} (y_k + \rho_k) \tag{4-40}$$

$$\text{for } j = 0, 1, \ldots, n \quad .$$

The difference between (4-40) and (4-39) is the error in the solution and is given by

$$\left|g_j - \tilde{g}_j\right| = \left|\frac{1}{\Delta}\sum_{k=o}^{n} \Delta_{kj}\rho_k\right| . \tag{4-41}$$

Letting $|\rho(t)| \leq \sigma$, the maximum error in approximating the integral (4-1) by (4-34), it follows that

$$|g_j - \tilde{g}_j| \leq \frac{\sigma}{|\Delta|}\sum_{k=o}^{n} |\Delta_{kj}| \tag{4-42}$$

or

$$|g_j - \tilde{g}_j| \leq \sigma B \tag{4-43}$$

where

$$B \triangleq \max \frac{1}{|\Delta|}\left\{\sum_{k=o}^{n} |\Delta_{kj}| \ : \ j = 0, 1, \ldots, n\right\} . \tag{4-44}$$

Estimates for the maximum integration error, σ, for the standard approximate quadratures are well known.[17] For example, for $o \leq t \leq b$,

$$\sigma \leq \frac{b}{2} h \max_{o \leq t \leq b} \left\{\frac{d}{d\tau}[x(t - \tau)g(\tau)]\right\} \tag{4-45}$$

for the rectangular rule,

$$\sigma \leq \frac{b}{12} h^2 \max_{o \leq t \leq b} \left\{\frac{d^2}{d\tau^2}[x(t - \tau)g(\tau)]\right\} \tag{4-46}$$

for the trapezoidal rule,

and

$$\sigma \leq \frac{b}{180} h^4 \max_{o \leq t \leq b} \left\{ \frac{d^4}{d\tau^4} \left[\kappa(t - \tau) g(\tau) \right] \right\} \tag{4-47}$$

for the Simpson rule.

Extending the Solution

Suppose the solution $g(t)$ of (4-1) has been computed or is otherwise known over some interval $[o,c]$. It is then possible to define the function

$$\mu(t) \triangleq \int_o^c x(t - \tau) g(\tau) d\tau \tag{4-48}$$

and to write (4-1) as

$$y(t) = \mu(t) + \int_c^t x(t - \tau) g(\tau)\, d\tau \tag{4-49}$$

$$\text{for } c \leq t \quad .$$

Since $\mu(t)$ is known, by (4-48), the problem reduces to determining $g(t)$ from the equation

$$y(t) - \mu(t) = \int_c^t x(t-\tau) g(\tau) d\tau \tag{4-50}$$

$$\text{for } t \geq c.$$

This method of extending the solution is most useful in the equation of the type*

*See Appendix II, equations (A2-12) and (A2-13).

$$\varphi(t) = f(t) + \lambda \int_0^t K(t - \tau)\varphi(\tau)d\tau \tag{4-51}$$

corresponding to (4-1).

If $\varphi(t)$ is known for $[o, c]$, the integral in (4-51) may be split into two integrals to become

$$\varphi(t) = f(t) + \lambda \int_0^c K(t - \tau)\, \varphi(\tau)d\tau + \lambda \int_c^t K(t - \tau)\varphi(\tau)d\tau \,. \tag{4-52}$$

Defining the function

$$v(t) \triangleq f(t) + \lambda \int_0^c K(t - \tau)\varphi(\tau)d\tau \;, \tag{4-53}$$

equation (4-51) may be written

$$\varphi(t) = v(t) + \lambda \int_c^t K(t - \tau)\, \varphi(\tau)d\tau \tag{4-54}$$

in terms of the new known function $v(t)$.

This decomposition, (4-53) and (4-54), is illustrated most simply by the familiar problem of determining the step response of a system when its response to a rectangular pulse of duration c and applied at $t = 0$ is known. For $t \leq c$ seconds the response is the step response since for that interval the pulse is a step. Proceeding from this point, with the solution $\varphi(t)$ known on the interval $[o, c]$, the solution is extended by (4-53) and (4-54) for values of t beyond this interval.

Reswick's Algorithm

Goodman and Reswick[11] approximate a system's impulse response by a sequence of impulses of variable weights and spacing. Convolving the input with this impulse train, they compare the result with the actual system output and adjust the weights and locations of the impulses until this error is tolerably small.

Performing these adjustments sequentially, the actual impulse response $g(t)$ is first approximated by a judiciously chosen impulse component and the output error for this model is computed. This error in $y(t)$ is attributed to convolving $x(t)$ with the error in approximating $g(t)$ by just one impulse. A second impulse is added to the model of $g(t)$ and the output error is again associated with a remanent component of $g(t)$.

In this way the error in the previous n-impulse-model for $g(t)$ is hopefully reduced by the addition of the proper $(n+1)^{st}$ impulse.

Each of the approximations of this chapter may be regarded as the approximation of the waveforms $g(t)$ and $x(t)$ by a sequence of uniformly spaced impulses whose weights are determined by the approximation employed. For example, the weights corresponding to the staircase approximations of Figure 1 and equations (4-3) are given by the solution (4-6).

Since only the samples of the waveforms are used in all of the methods of this chapter it is clear that sampled-data system analogues and Z transform methods of sampling and interpolating systems apply.[18]

CHAPTER V

EXAMPLES AND DISCUSSION

To illustrate the performance of the inversion methods presented in this thesis, consider solving

$$y(t) = \int_0^t x(t-\tau)\ g(\tau)\ d\tau \tag{5-1}$$

by the methods of Chapters II, III and IV, when it is given that

$$x(t) = e^{-at} \tag{5-2}$$

and

$$y(t) = t\,e^{-bt}\ . \tag{5-3}$$

<u>The Division Algorithm</u>

Differentiating (5-2) to get

$$k(t) = -\ a\,e^{-at}\ , \tag{5-4}$$

it follows that

$$k_2(t) = \int_0^t k(t-\tau)\ k(\tau)\ d\tau = a^2 e^{-at}\ t. \tag{5-5}$$

Similarly,

$$k_3(t) = \int_0^t k(t-\tau)\ k_2(\tau)\,d\tau = -a^3\,e^{-at}\frac{t^2}{2} \tag{5-6}$$

and

$$k_4(t) = \int_o^t k(t-\tau)\ k_3(\tau)d\tau = a^4\ e^{-at}\ \frac{t^3}{3!}\ . \tag{5-7}$$

The pattern is clear and an easy induction shows that

$$k_n(t) = (-1)^n\ a^n\ e^{-at}\ \frac{t^{n-1}}{(n-1)!} \tag{5-8}$$

for $n = 2,3,\ldots$.

Thus, from equation (2-38), since

$$x(o) = 1\ , \tag{5-9}$$

$$r(t) = k(t) - \frac{1}{x(o)}\ k_2(t) + \left(\frac{1}{x(o)}\right)^2 k_3(t) + \ldots + \left(\frac{-1}{x(o)}\right)^n k_n(t) + \ldots$$

becomes

$$r(t) = -ae^{-at} - a^2e^{-at}t - a^3e^{-at}\ \frac{t^2}{2} - \ldots - a^{n+1}\ e^{-at}\ \frac{t^n}{n!} - \ldots \tag{5-10}$$

Factoring a $\left\{-ae^{-at}\right\}$,

$$r(t) = -ae^{-at}\left[1 + at + a^2\ \frac{t^2}{2} + \ldots + \frac{a^n t^n}{n!} + \ldots\right] \tag{5-11}$$

whence

$$r(t) = -ae^{-at}\ [e^{at}] \tag{5-12}$$

and

$$r(t) = -a \quad \text{for} \quad t \geq 0 . \tag{5-13}$$

Substituting for $r(t)$ in equation (2-66),

$$g(t) = \frac{1}{x(o)}\left[y'(t) - \frac{1}{x(o)}\int_o^t r(t-\tau)\, y'(\tau)\, d\tau\right] \tag{5-14}$$

and computing

$$y'(t) = e^{-bt}(1-bt) \tag{5-15}$$

from (5-3), the solution appears as

$$g(t) = e^{-bt}(1-bt) - \int_o^t (-a)\ \mu_{-1}(t-\tau)[e^{-b\tau}(1-b\tau)]\, d\tau \tag{5-16}$$

where $\mu_{-1}(t)$ is the unit step.

Thus

$$\begin{aligned} g(t) &= e^{-bt}(1-bt) + a\int_o^t \frac{d}{d\tau}[e^{-b\tau}]\, d\tau \\ &= e^{-bt}(1-bt) + a\, t\, e^{-bt} \end{aligned}$$

and the desired solution is

$$g(t) = [1 - (b-a)t]e^{-bt} \tag{5-17}$$

as is easily checked.

The Factor Algorithm

Since $x(t)$ and $y(t)$ in this example have derivatives of all orders on $[o,t]$, the derivatives of $g(t)$ at $t = 0$ may be computed by equations (3-8) and (3-9),

$$g(o) = \frac{y'(o)}{x(o)} ,$$

$$g^{(n)}(o) = \frac{1}{x(o)}\left[y^{(n+1)}(o) - \sum_{j=1}^{n} x^{(j)}(o) g^{(n-j)}(o) \right] \qquad (5\text{-}18)$$

for $n = 1,2,\ldots$.

For $y^{(n)}(t)$,

$$y'(t) = e^{-bt}(1 - bt) ,$$

$$y''(t) = -b\, e^{-bt}(2 - bt) ,$$

$$y'''(t) = b^2\, e^{-bt}(3 - bt)$$

and, by induction,

$$y^{(n)}(t) = (-1)^{n-1}\, b^{n-1}\, e^{-bt}\, (n - bt) \qquad (5\text{-}19)$$

for $n = 0,1,2,\ldots$.

Similarly,

$$x'(t) = -a\, e^{-at}$$

$$x''(t) = a^2\, e^{-at}$$

$$x'''(t) = -a^3\, e^{-at}$$

and

$$x^{(n)}(t) = (-1)^n a^n e^{-at} \tag{5-20}$$

$$\text{for} \quad n = 0, 1, 2, \ldots \quad .$$

Thus, setting $t = 0$ in (5-19) and (5-20)

$$y^{(n+1)}(o) = (-1)^n b^n (n+1) \tag{5-21}$$

and

$$x^{(j)}(o) = (-1)^j a^j \; . \tag{5-22}$$

The recursive process, (5-18), is now begun.

$$g(o) = \frac{y'(o)}{x(o)} = \frac{1}{1} = 1 \; ,$$

$$g'(o) = \frac{1}{x(o)} [y''(o) - x'(o)\, g(o)]$$

$$= \frac{1}{1} [-2b - (-a)(1)]$$

$$= a - 2b \; ,$$

$$g''(o) = \frac{1}{x(o)} [y^{(3)}(o) - (x'(o) g'(o) + x^{(2)}(o)\, g(o)]$$

$$= \frac{1}{1} [3b^2 - (-a[a - 2b] + a^2[1])]$$

$$= b(3b - 2a) \; ,$$

$$g^{(3)}(o) = \frac{1}{x(o)}\,[y^{(4)}(o) - (x'(o)g^{(2)}(o) + x^{(2)}(o)\ g'(o)$$

$$+ x^{(3)}(o)\ g(o))]$$

$$= \frac{1}{1}\,[-4b^3 - (-a[b(3b-2a)] + a^2[a-2b] + [-a^3][1])]$$

$$= -b^2(4b - 3a)\ .$$

If the guess is now made that

$$g^{(n)}(o) = (-1)^n b^{n-1}[(n+1)b - na] \tag{5-23}$$

for $n = 0,1,2,\ldots$,

a simple but lengthy induction argument establishes its validity.

By virtue of (5-23), it follows that

$$\frac{g^{(n)}(o)t^n}{n!} = \frac{1}{n!}\,(-bt)^n + \frac{n}{n!}\,(1 - \frac{a}{b})(-bt)^n \tag{5-24}$$

and, since

$$e^{-bt} = \sum_{n=o}^{\infty} \frac{1}{n!}\,(-bt)^n\ , \tag{5-25}$$

the convergence being uniform and absolute, it also follows that

$$(-b)e^{-bt} = \sum_{n=1}^{\infty} \frac{n}{n!}\,(-bt)^{n-1}\,(-b) \tag{5-26}$$

and that

$$t(-be^{-bt}) = -1 + \sum_{n=o}^{\infty} \frac{n}{n!} (-bt)^n \ . \qquad (5\text{-}27)$$

The Taylor series expansion of $g(t)$ about the origin may now be summed to give $g(t)$ in closed form:

$$g(t) = \sum_{n=o}^{\infty} \frac{g^{(n)}(o)}{n!} t^n$$

$$= g(o) + \sum_{n=1}^{\infty} \frac{g^{(n)}(o)}{n!} t^n$$

$$= 1 + (e^{-bt} - 1) + (1 - \tfrac{a}{b}) t(-be^{-bt}) \ ,$$

or, finally,

$$g(t) = [1 - (b - a)t] \ e^{-bt} \ .$$

The Real Time Inversions

All of the real time inversion schemes of Chapter IV are easily programmed and compared on a digital computer.

For the problem just considered, for example, setting $a = 2$, $b = 1$ and choosing $h = 0.01$ and $\varepsilon = 0.001$, the approximate solutions of systems (4-25), (4-6), (4-10), (4-12), (4-28) and (4-33) approximate the solution and are listed in the order of increasing accuracy.

As an illustration, Figure 14 shows the approximation to $g(t)$ which results when the trapezoidal rule inversion system (4-28) is used with the parameters given. This approximation differs from the exact

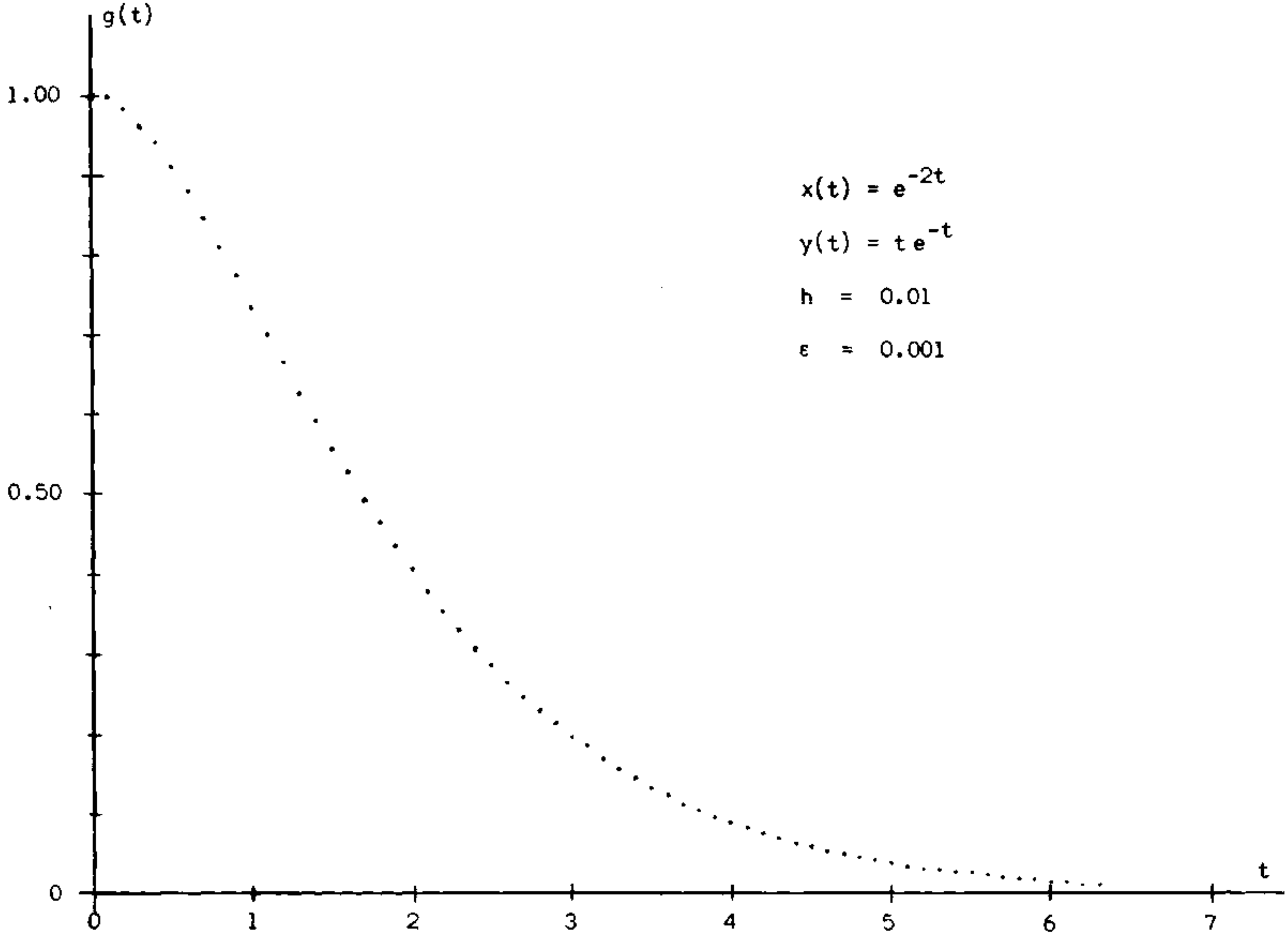

Figure 14. The Computation of $g(t)$ by Equations (4-28).

solution by less than one tenth of one percent while the solution by the staircase approximation, (4-6), for the same parameters, has a maximum error of about one and one half percent.

If the sampling rate is decreased by a factor of ten, so that $h = 0.10$, the error in the trapezoidal solution increases to only two fifths of one percent. Since the time constant of $x(t)$ is 0.50 seconds, this sampling rate is quite crude, being only 5 samples per time constant, yet the trapezoidal solution still provides a useful approximation to $g(t)$.

If the special case $a = 0$, $b = 0$ is considered, all of the systems yield the exact answer since all of the underlying approximations become exact for this case.

To see this and to further illustrate the theory, consider the solution of system (4-25). Since

$$x(t) = 1 \quad \text{for} \quad t \geq 0 \ ,$$

and

$$y(t) = t \quad \text{for} \quad t \geq 0 \ ,$$

it follows that

$$g_n = \frac{1}{1}\left[\frac{(n+1)h}{h} - \sum_{j=o}^{n-1} g_j\right]$$

$$= (n+1) - (g_o + g_1 + \dots + g_{n-1}) \ .$$

But

$$g_o = \frac{1}{1}\frac{h}{h} = 1 ,$$

$$g_1 = \frac{1}{1}[\frac{2h}{h} - (1)(1)] = 1 ,$$

$$g_2 = \frac{1}{1}[\frac{3h}{h} - (1)(1) - (1)(1)] = 1 ,$$

so that

$$g_n = (n + 1) - n = 1$$

independent of h.

That

$$g(t) = 1 \quad \text{for} \quad t \geq 0$$

when $x(t) = 1$ and $y(t) = t$ is easily checked.

As another simple yet informative example, suppose

$$x(t) \equiv y(t)$$

and consider the solution of system (4-25).

Since $x_i = y_i$ for $i = 0,1,2,\ldots,$

$$g_o = \frac{1}{x_1}[\frac{x_1}{h}] = \frac{1}{h}$$

$$g_1 = \frac{1}{x_1}[\frac{x_2}{h} - (x_2)\frac{1}{h}] = 0$$

$$g_2 = \frac{1}{x_1}[\frac{x_3}{h} - (x_3)(\frac{1}{h}) - (x_2)(0)] = 0 ,$$

$$\vdots$$

(continued)

$$g_n = \frac{1}{x_1}\left[\frac{x_{n+1}}{h} - (x_{n+1})(\frac{1}{h}) - 0 - \dots - 0\right] = 0$$

and the solution,

$$g_n = \begin{cases} \frac{1}{h} & \text{for } n = 0 \\ 0 & \text{for } n = 1,2,\dots , \end{cases}$$

reveals that $g(t)$ must be an "impulse" in order to yield $x(t)$ when convolved with $x(t)$.

Some Simple Cases

1. If $x(t) = 1$ for $t \geq 0$, the unit step, the division algorithm simplifies.

In fact

$$g(t) = \left\{x(o)\ \delta(t) + x'(t)\right\} \overline{\big|\big|\ y'(t)}$$

becomes

$$g(t) = \left\{1\delta(t) + 0\right\} \overline{\big|\big|\ y'(t)}$$

$$= y'(t)$$

which shows that the impulse response of a system is the derivative of the step response.

2. If $x(t) = 1 + t$ and $y(t) = t$

for $t \geq 0$,

the division algorithm yields

$$g(t) = \left\{1\ \delta(t) + 1\right\} \overline{\big|\big|\ 1}$$

which when carried out, gives

$$
\begin{array}{l|l}
 & 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots \\ \hline
\left\{\delta(t) + 1\right\} & 1 \\
 & \underline{1 + t} \\
 & \quad - t \\
 & \quad \underline{- t - \frac{t^2}{2}} \\
 & \qquad + \frac{t^2}{2} \\
 & \qquad \underline{+ \frac{t^2}{2} + \frac{t^3}{3!}} \\
 & \qquad\qquad - \frac{t^3}{3!} \\
 & \qquad\qquad\ \ \vdots
\end{array}
$$

Thus, summing the resulting series,

$$g(t) = e^{-t}$$

which is easily verified.

3. If

$$x(t) = \cos t \quad \text{and} \quad y(t) = \sin t$$

the construction of the function $r(t)$ by the infinite expansion of equation (2-38) is quite difficult.

By the method of Evans presented in Appendix IV, however, this $x(t)$ may be recognized as the solution of a simple differential equation and a corresponding differential equation may be constructed whose solution is $r(t)$. Example (ii) of Appendix IV derives $r(t)$ for

$$x(t) = A \sin \omega t + B \cos \omega t$$

by this method and constructs the differential equation

$$r''(t) + \frac{A}{B} \omega \, r'(t) = 0$$

$$r(0) = \omega A$$

$$r'(0) = -\frac{\omega^2}{B} (A^2 + B^2)$$

whose solution is

$$r(t) = \omega A - \frac{\omega}{A} (A^2 + B^2) \left[1 - e^{-\frac{A}{B} \omega t} \right]$$

$$\text{for } A \neq 0 ,$$

and

$$r(t) = -\omega^2 B \, t$$

$$\text{for } A = 0 .$$

Setting $A = 0$, $B = 1$ and $\omega = 1$ in these results, the $r(t)$ corresponding to $x(t) = \cos t$ is seen to be

$$r(t) = - t .$$

The solution for $g(t)$ is thus given by (2-66) as

$$g(t) = \frac{1}{x(o)} \left[y'(t) - \frac{1}{x(o)} \int_o^t r(t-\tau) \, y'(\tau) \, d\tau \right]$$

$$= \frac{1}{1} \left[\cos t - \frac{1}{1} \int_0^t (t-\tau) \cos \tau \, d\tau \right]$$

$$= \cos t - (\cos t - 1)$$

$$= 1 \; .$$

That

$$g(t) = 1$$

is easily checked.

Example (i) of Appendix IV derives $r(t)$ by this method for the example

$$x(t) = e^{-at}$$

and

$$y(t) = t\, e^{-bt}$$

and gets

$$r(t) = -a$$

just as in equation (5-13).

Discussion

In both of the methods of Chapter III and Chapter IV the solutions for $g(t)$ in terms of $x(t)$ and $y(t)$ were given recursively.

Comparing, for example, the equations (3-38) and (4-3), a convolution summation of the type

$$a_n = \sum_{j=o}^{n} b_j \, c_{n-j} \tag{5-28}$$

for $n = 0,1,2,\ldots$

was to be solved for the $\{c_i\}$ in terms of the $\{a_i\}$ and the $\{b_i\}$. The solution was shown to be of the form

$$c_o = \frac{a_o}{b_o} \tag{5-29}$$

$$c_n = \frac{1}{b_o}\left[a_n - \sum_{j=o}^{n-1} c_j b_{n-j} \right] \tag{5-30}$$

for $n = 1,2,\ldots$.

Now by comparing equations (5-28) and (5-29) and (5-30) with Theorem (A1-XII) and Theorem (A1-XIII) in Appendix I, it is recognized that the unknowns $\{c_i\}$ may be found by the familiar long division algorithm applied to the quotient of the two power series

$$\sum_{o}^{\infty} a_n z^n \quad \text{and} \quad \sum_{o}^{\infty} b_n z^n \, .$$

In fact, equation (5-28) may be written in terms of power series as the product

$$\sum_{o}^{\infty} a_n z^n = \left(\sum_{o}^{\infty} b_n z^n \right)\left(\sum_{o}^{\infty} c_n z^n \right) \tag{5-31}$$

and (5-29) and (5-30) as the quotient

$$\sum_{o}^{\infty} c_n z^n = \frac{\sum_{o}^{\infty} a_n z^n}{\sum_{o}^{\infty} b_n z^n} \tag{5-32}$$

provided $b_o \neq 0$.

Fielder[(19)] has tabulated the first twelve coefficients which arise when operations such as the division of (5-32) are performed on Taylor series and his results permit the direct calculation of c_{12}, for example, from the values $a_o,\ldots,a_{12}$ and $b_o,\ldots,b_{12}$ without first calculating $c_o,c_1,\ldots,c_{11}$ as in the recursive process (5-30).

Thus it is especially interesting to observe that all three of the inversions developed in this thesis embody as an integral part of their performance the familiar steps of the long division algorithm.

While the exposition of Chapter II use Laplace Transforms in deriving the inversion series, (2-17), the identification of this series with the Neumann series in Appendix I reveals that the division algorithm for the inversion of convolution is not dependent upon the applicability of the Laplace Transformation.

In fact, if $x(t)$ were such that $\mathcal{L}\{x(t)\}$ does not exist, for example,

$$x(t) = e^{t^2},$$

the method of Chapter II still applies* for, $x(t)$ is continuously

---

*The inversions of Chapter III and Chapter IV of course still apply since they do not involve Laplace Transforms at all.

differentiable on any finite interval $[o,t]$. Since $x(t)$ increases faster than the convergence factor $e^{-\sigma t}$ of the Laplace Transform, the transform does not exist for this function.

This example illustrates a severe limitation of the frequency domain inversion and the greater power of the time domain techniques presented in this research which do not depend on the future behavior of the functions $x(t)$ and $y(t)$ for the computation of $g(t)$.

It was indicated in Chapter I that Zame's feedback inversion system, Figure 6, is only a formal procedure and that the system may not be realizable. This may be seen by considering equation (1-23) which is of the form

$$H(s) = a \frac{1}{1 - (1 - aX(s))} , \tag{5-33}$$

or

$$H(s) = a \frac{1}{1 - \tilde{K}(s)} \tag{5-34}$$

where

$$\tilde{K}(s) \triangleq 1 - aX(s) . \tag{5-35}$$

Expanding (5-34) formally, as was done in (2-22),

$$H(s) = a \left[1 + [\tilde{K}(s)] + [\tilde{K}(s)]^2 + \ldots + [\tilde{K}(s)]^n + \ldots \right] \tag{5-36}$$

and inverting to get a time domain expansion analogous to equation (2-24), it follows that

$$h(t) = a\,[1 + \tilde{k}(t) + \tilde{k}(t) \otimes \tilde{k}(t) + \ldots + \tilde{k}(t) \otimes \tilde{k}(t) \otimes \ldots \otimes \tilde{k}(t) + \ldots] \tag{5-37}$$

where

$$\tilde{k}(t) = \mathscr{L}^{-1}[\tilde{K}(s)]. \tag{5-38}$$

Now

$$\tilde{k}(t) = \delta(t) - a\,x(t) \tag{5-39}$$

and because of the impulsive component in $\tilde{k}(t)$, each term of the expansion (5-37) contains a unit impulse -- the n^{th} term being the n-fold self convolution of $\tilde{k}(t)$.

Thus, the expansion (5-37) contains an infinite series of impulses and is neither a useful nor a valid expansion.

It is also of interest to observe that precisely the expansions of Chapter II are obtained when the convolution integral (2-1) is either differentiated, or integrated by parts, to get equations (A2-12) or (A2-13), and then the Laplace Transform of these equations is evaluated.

Davis's[20] method of solving integral equations by infinite series of operators is illustrated in Appendix IV by Theorem (A4-I) and an example which constructs the resolvent kernel $r(t)$ by means of a "resolvent generatrix."

This method of using a reciprocal series to generate the desired coefficients, as in (A4-74), was first developed by N. Sonine[21], in 1884, and bears a remarkable resemblance to the operational methods of the theory of Z transforms.

Conclusion

Three distinct time domain inversions for the operation of convolution have been developed in this thesis. For each inversion scheme an algorithm has been devised to aid in the visualization of the process. Each inversion scheme has been shown to embody in a fundamental way, the familiar ordinary long division algorithm of arithmetic.

Being time domain techniques, each of the inversion schemes is applicable to a wider class of problems than the frequency domain inversion by Laplace Transforms. Moreover, the practical, real time inversions are computationally far simpler than the Laplace Transform method and can be performed "on line."

It is hoped that the combination of the three inversion algorithms will be of as much help in inverting convolution as the graphical algorithm of Figure 2 is in performing it.

APPENDICES

APPENDIX I

SOME IMPORTANT THEOREMS OF MATHEMATICAL ANALYSIS

Integrals Depending on a Parameter

Theorem (Al-I) Continuity[22]

Let f be continuous at each point (x,y) of a rectangle

$$R = \left\{(x,y) : a \leq x \leq b, \quad c \leq y \leq d\right\} .$$

Assume that α is of bounded variation on $[a,b]$ and let F be the function defined on $[c,d]$ by the equation

$$F(y) = \int_a^b f(x,y)\, d\alpha(x) .$$

Then F is continuous on $[c,d]$.

In other words, if $y_o \in [c,d]$, then

$$\lim_{y \to y_o} \int_a^b f(x,y)\, d\alpha(x) = \int_a^b \lim_{y \to y_o} f(x,y)\, d\alpha(x)$$

$$= \int_a^b f(x,y_o)\, d\alpha(x) .$$

Corollary of Theorem (Al-I)[23]

If f is continuous on the rectangle R of Theorem I and if g is Riemann integrable on $[a,b]$, then the function F defined by the equation

$$F(y) = \int_a^b g(x)\ f(x,y)\ dx$$

is continuous on $[c,d]$. That is, if $y_o \varepsilon [c,d]$,

$$\lim_{y \to y_o} \int_a^b g(x)\ f(x,y)\ dx = \int_a^b g(x)\ f(x,y_o)\ dx\ .$$

Theorem (Al-II) Leibniz's Rule for Differentiation(24)

Let f and $D_2 f$ be continuous on the rectangle R of Theorem I.

Let p and q be two functions having finite derivatives p' and q' on $[c,d]$ and assume that $a \leq p(y) \leq b$ and that $a \leq q(y) \leq b$ for each y in $[c,d]$.

Define F by the equation

$$F(y) = \int_{p(y)}^{q(y)} f(x,y)dx \quad \text{if} \quad y\ \varepsilon\ [c,d]\ .$$

Then $F'(y)$ exists for each y in (c,d) and is given by

$$F'(y) = \int_{p(y)}^{q(y)} D_2 f(x,y)dx + f[q(y),y]\ q'(y) - f[p(y),y]p'(y)\ .$$

Integration by Parts

Theorem (Al-III)(25)

If f is Riemann-Stieltjes integrable with respect to the integrator α on $[a,b]$ then α is Riemann-Stieltjes integrable with respect to the integrator f on $[a,b]$ and

$$\int_a^b f(x)\, d\alpha(x) + \int_a^b \alpha(x)\, df(x) = f(x)\,\alpha(x)\Big|_a^b .$$

Double Integrals and Iterated Integrals

Theorem (A1-IV)[26]

Let f be integrable over the rectangle R of Theorem I and suppose the iterated integral

$$F(y) = \int_a^b f(x,y)\, dx, \quad \text{for} \quad y \in [c,d], \quad \text{exists.}$$

Then F is integrable on $[c,d]$ and

$$\iint_R f(x,y)\, dA = \int_c^d F(y)\, dy = \int_c^d \left\{\int_a^b f(x,y)dx\right\} dy .$$

Theorem (A1-V)[27]

If f is continuous on the rectangle R of Theorem I, then

$$\iint_R f(x,y)\, dA = \int_c^d \left\{\int_a^b f(x,y)\, dx\right\} dy$$
$$= \int_a^b \left\{\int_c^d f(x,y)\, dy\right\} dx .$$

The Convolution Integral

Definition (A1-I)[28]

Given two functions f and g both absolutely integrable on $(-\infty,\infty)$, let S denote the set of t for which the improper integral

$$h(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

converges. This integral defines a function h on S called the convolution of f and g.

<u>Theorem (A1-VI)</u>[29]

If f and g are absolutely integrable on $(-\infty,\infty)$ and if either f or g is bounded on $(-\infty,\infty)$, then the convolution integral

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

converges absolutely for all t in $(-\infty,\infty)$.

<u>Theorem (A1-VII)</u>[30]

If f and g are absolutely integrable on $(-\infty,\infty)$, and if both integrals

$$\int_{-\infty}^{\infty} |f(\tau)|^2\, d\tau \quad \text{and} \quad \int_{-\infty}^{\infty} |g(\tau)|^2\, d\tau$$

converge, then the convolution integral

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

converges absolutely for every t in $(-\infty,\infty)$.

<u>Theorem (A1-VIII)</u>[31]

Let f and g be absolutely integrable on $(-\infty,\infty)$ and assume that either

(i) at least one of f or g is bounded on $(-\infty,\infty)$, or

(ii) both integrals $\int_{-\infty}^{\infty} |f(\tau)|^2 \, d\tau$ and $\int_{-\infty}^{\infty} |g(\tau)|^2 \, d\tau$ converge.

Then the convolution integral

$$h(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

exists for every t in $(-\infty,\infty)$ and has the following properties:

(a) h is bounded on $(-\infty,\infty)$,

(b) $\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$ converges uniformly on $(-\infty,\infty)$,

(c) h is continuous and h is absolutely integrable on $(-\infty,\infty)$ if at least one of f or g is continuous on $(-\infty,\infty)$.

Theorem (Al-IX) Theorem of Titchmarsh[32]

If the convolution of two functions f and g, both continuous on $[0,\infty)$ is identically equal to zero, then at least one of the functions f or g is identically zero.

Theorem (Al-X) Cauchy's Formula[33]

If f is integrable on $[a,b]$, then

$$\underbrace{\int_a^x \cdots \int_a^x}_{\text{n times}} f(x) \underbrace{dx \ldots dx}_{\text{n times}} = \frac{1}{(n-1)!} \int_a^x (x-\varphi)^{n-1} f(\varphi)\, d\varphi$$

for $x \in [a,b]$.

Definition (A1-II)[34]

The discontinuities of a function $K(t,\tau)$ are said to be regularly distributed in a region S if they all lie on a finite number of curves with continuously turning tangents no one of which is met by a line parallel to either coordinate axis in more than a finite number of points.

Theorem (A1-XI)[35]

If the two functions $\varphi(t,\tau)$ and $\Psi(t,\tau)$ are finite in the rectangle R of Theorem (A1-I) and if their discontinuities, if they have any, are regularly distributed, the function

$$F(t,\tau) = \int_a^b \Phi(t,\varphi)\, \Psi(\varphi,\tau)\, d\varphi$$

is continuous throughout R.

Infinite Series

Theorem (A1-XII)[36]

Let $\sum_o^\infty u_n$ and $\sum_o^\infty v_n$ converge absolutely to U and V respectively.

Define

$$w_o = u_o v_o$$

and

$$w_n = \sum_{j=o}^{n} u_j v_{n-j} \, .$$

Then the series $\sum_{o}^{\infty} w_n$ is absolutely convergent and $\sum_{o}^{\infty} w_n = UV$.

Theorem (A1-XIII)[37]

Consider a function defined as the quotient of two power series

$$f(z) = \frac{a_o + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots}{b_o + b_1 z + b_2 z^2 + \ldots b_n z^n + \ldots}$$

where $b_o \neq 0$, and where both of the series are convergent in some interval $|z| < R$.

Then, for sufficiently small values of z, the function f can be represented as a power series

$$f(z) = c_o + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots$$

whose coefficients may be found by the process of long division, or, what is equivalent, by solving the relations

$$b_o c_o = a_o$$

$$b_o c_1 + b_1 c_o = a_1$$

$$\ldots\ldots\ldots\ldots$$

$$b_o c_n + b_1 c_{n-1} + \ldots + b_n c_o = a_n$$

$$\ldots\ldots\ldots\ldots$$

successively for $c_o, c_1, c_2, \ldots$.

APPENDIX II

CONVOLUTION AS AN INTEGRAL EQUATION

Classification of Linear Integral Equations

In the theory of linear integral equations the following classification[38] is made:

$$\varphi(t) - \lambda \int_a^t K(t,\tau)\, \varphi(\tau)\, d\tau = f(t) \qquad \text{(A2-1)}$$

for $t \in [a, b]$

is a <u>Volterra equation of the second kind</u>,

$$\varphi(t) - \lambda \int_a^b K(t,\tau)\, \varphi(\tau)\, d\tau = f(t) \qquad \text{(A2-2)}$$

for $t \in [a, b]$

is a <u>Fredholm equation of the second kind</u>,

$$f(t) = \int_a^t K(t,\tau)\, \varphi(\tau)\, d\tau \qquad \text{(A2-3)}$$

for $t \in [a, b]$

is a <u>Volterra equation of the first kind</u> and

$$f(t) = \int_a^b K(t,\tau)\, \varphi(\tau)\, d\tau \qquad \text{(A2-4)}$$

for $t \in [a, b]$

is a Fredholm equation of the first kind.

In all of these equations, the function $\varphi(t)$ is unknown, the functions $f(t)$ and $K(t, \tau)$ are assumed to be given, and λ is a number, called the parameter of the equation.

The function $K(t,\tau)$ is called the kernel of the equation and is defined in the (t,τ) - plane in the closed square

$$[a,b] \times [a,b] \triangleq \{(t,\tau): a \leq t \leq b, \quad a \leq \tau \leq b\} \quad ,$$

hereafter called the basic square.

The closed interval $[a,b]$ is called the basic interval.

Changing the Type of an Equation

In Appendix III it is shown how a Volterra equation may be treated as a special case of a Fredholm equation, and how Fredholm equations of the second kind may be solved by the Neumann series of successive approximations.

Volterra equations of the first kind may be transformed into corresponding equations of the second kind in two ways.[39]

Differentiating both sides of (A2-3), assuming $K(t,t) \neq 0$,

$$\varphi(t) + \int_a^t H(t,\tau)\, \varphi(\tau)\, d\tau = \frac{f'(t)}{K(t,t)} \quad , \tag{A2-5}$$

where

$$H(t,\tau) \triangleq \frac{1}{K(t,t)} \frac{\partial}{\partial t} K(t,\tau), \tag{A2-6}$$

which is an equation of the second type.

Alternatively, setting

$$\omega(t) = \int_a^t \varphi(\tau)\, d\tau \qquad \text{(A2-7)}$$

so that

$$\varphi(t) = \omega'(t) , \qquad \text{(A2-8)}$$

an integration by parts of (A2-3) yields

$$\omega(t) - \int_a^t Q(t,\tau)\, \omega(\tau)\, d\tau = \frac{f(t)}{K(t,t)} , \qquad \text{(A2-9)}$$

where

$$Q(t,\tau) \triangleq \frac{1}{K(t,t)} \frac{\partial}{\partial\tau} K(t,\tau) , \qquad \text{(A2-10)}$$

and (A2-9) is also an equation of second kind.

Convolution

When these techniques are applied to the convolution integral

$$y(t) = \int_o^t x(t-\tau)\, g(\tau) d\tau , \qquad \text{(A2-11)}$$

whose kernel is $x(t-\tau)$ of the closed cycle type[40], the equivalent Volterra equations of the second kind are

$$g(t) = \frac{y'(t)}{x(o)} + \left(-\frac{1}{x(o)}\right) \int_o^t x'(t-\tau)\, g(\tau)\, d\tau \qquad \text{(A2-12)}$$

and

$$A(t) = \frac{y(t)}{x(o)} + \left(- \frac{1}{x(o)}\right) \int_o^t x'(t-\tau)\; A(\tau)\, d\tau \qquad \text{(A2-13)}$$

where

$$g(t) = A'(t) \;. \qquad \text{(A2-14)}$$

Convolution and Linear Differential Equations

The solution of the differential equation

$$\alpha_o(t) \frac{d^n \mu}{dt^n} + \alpha_1(t) \frac{d^{n-1}\mu}{dt^{n-1}} + \ldots + \alpha_n(t)\,\mu = F(t) \qquad \text{(A2-15)}$$

with continuous coefficients together with the initial conditions

$$\mu(o) = c_o, \;\; \mu'(o) = c_1, \ldots, \mu^{(n-1)}(o) = c_{n-1} \qquad \text{(A2-16)}$$

can be reduced by successive integrations to the solution of a corresponding Volterra integral equation of the second kind

$$\alpha_o(t)\; \varphi(t) + \int_o^t G(t,\tau)\; \varphi(\tau)\; d\tau = f(t) \;. \qquad \text{(A2-17)}$$

The equivalence of the initial value problem (A2-15), (A2-16) and the integral equation (A2-17) is established by the identification[41]

$$G(t,\tau) = \sum_{i=1}^{n} \alpha_i(t) \frac{(t - \tau)^{i-1}}{(i - 1)!}, \qquad \text{(A2-18)}$$

$$f(t) = F(t) - c_{n-1}\alpha_1(t) - (C_{n-1}t + c_{n-2})\,\alpha_2(t) - \dots$$
$$- (c_{n-1}\frac{t^{n-1}}{(n-1)!} + \dots + c_1 t + c_o)\alpha_n(t) \;, \tag{A2-19}$$

and

$$\varphi(t) = \frac{d^n\mu}{dt^n} \;. \tag{A2-20}$$

If the differential equation (A2-15) has constant coefficients, equation (A2-18) shows that the kernel $G(t,\tau)$ of the integral equation (A2-17) will be a polynomial in $(t - \tau)$ and hence of the closed cycle type. Time-invariant linear systems are described by differential equations with constant coefficients and the equivalent Volterra integral equation for such a system is thus a convolution type equation.

If the physical system described by (A2-15) is initially at rest, or relaxed, the initial condition terms of (A2-16) are all zero and

$$f(t) = F(t) \;.$$

In this case it is also possible to express the solution of (A2-15) as

$$\mu(t) = \int_o^t K(t,\tau)\, F(\tau)\, d\tau$$
$$\mu^{(j)}(0) = 0, \quad j = 0,1,\dots,n-1 \tag{A2-21}$$

where $K(t,\tau)$ is the <u>one-sided Green's function</u> of the linear differential operator N defined by

$$N \triangleq a_o(t)\,\frac{d^n}{dt^n} + a_1(t)\,\frac{d^{n-1}}{dt^{n-1}} + \dots + a_n(t) \;. \tag{A2-22}$$

Here the kernel $K(t,\tau)$ is computed from

$$K(t,\tau) = \frac{(-1)^{n-1}}{a_o(\tau)\;W(\tau)} \begin{vmatrix} \varphi_1(t) & \varphi_2(t) & \dots & \varphi_n(t) \\ \varphi_1(\tau) & \varphi_2(\tau) & \dots & \varphi_n(\tau) \\ \varphi_1'(\tau) & \varphi_2'(\tau) & \dots & \varphi_n'(\tau) \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)}(\tau) & \varphi_2^{(n-2)}(\tau) & \dots & \varphi_n^{(n-2)}(\tau) \end{vmatrix} \tag{A2-23}$$

where $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ are n linearly independent solutions of $N u = 0$ and $W(t)$ is their Wronskian.

For the case of constant coefficients this kernel $K(t,\tau)$ is also a function only of the difference of the arguments and (A2-21) becomes

$$\mu(t) = \int_o^t K(t - \tau)\; F(\tau)\; d\tau \;, \tag{A2-24}$$

the convolution integral.

APPENDIX III

SUCCESSIVE APPROXIMATIONS AND THE NEUMANN SERIES

The Volterra Equation as a Fredholm Equation

The Volterra equation of the second kind

$$\varphi(t) = f(t) + \lambda \int_a^t K(t,\tau)\ \varphi(\tau)\ d\tau \tag{A3-1}$$

$$\text{for} \quad a \le t \le b,$$

may be written as a Fredholm equation of the second kind

$$\varphi(t) = f(t) + \lambda \int_a^b G(t,\tau)\ \varphi(\tau)\ d\tau \tag{A3-2}$$

$$\text{for} \quad a \le t \le b\ ,$$

by defining the kernel $G(t,\tau)$ as

$$G(t,\tau) \triangleq \begin{cases} K(t,\tau) & \text{for} \quad a \le \tau \le t \le b \\ 0 & \text{for} \quad a \le t < \tau \le b\ . \end{cases} \tag{A3-3}$$

The Fredholm kernel $G(t,\tau)$ is thus seen to vanish for $t < \tau$ and to equal the Volterra kernel $K(t,\tau)$ for $t \ge \tau$. This is illustrated in Figure 15 in the t,τ plane.

Such a transformation is convenient because of the great body of literature devoted to the Fredholm equation of the second kind and the relative scarcity of Volterra theory.

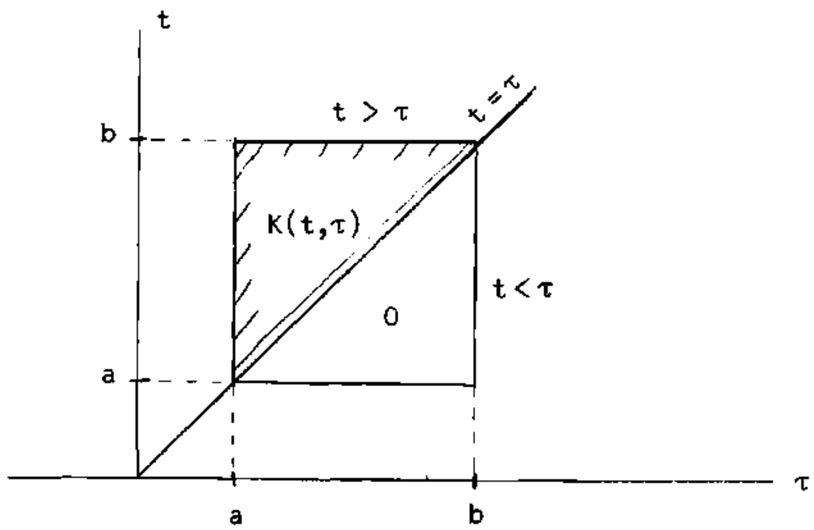

Figure 15. $G(t,\tau)$ Defined on the Basic Square.

The Method of Successive Approximations*

Beginning with an arbitrary initial approximation, $\varphi_0(t)$, from the class of functions under consideration, the iterative scheme

$$\varphi_{n+1}(t) = f(t) + \lambda \int_a^b G(t,\tau)\ \varphi_n(\tau)\ d\tau \tag{A3-4}$$

for $n = 0,1,2,\ldots$

may be defined.

Under the proper hypotheses the sequence of functions $\{\varphi_n(t)\}$ generated by equation (A3-4) may be shown to converge to the unique solution of equation (A3-2).

Introducing the operator notation

---

*The notation and terminology of this presentation are those of Mikhlin Ref. (43) and Ref. (46).

$$\mathcal{K}\,\varphi(t) \triangleq \int_a^b G(t,\tau)\,\varphi(\tau)\,d\tau, \tag{A3-5}$$

equation (A3-2) becomes

$$\varphi = f + \lambda\mathcal{K}\,\varphi \tag{A3-6}$$

and (A3-4) becomes

$$\varphi_{n+1} = f + \lambda\mathcal{K}\,\varphi_n \tag{A3-7}$$

for $n = 0,1,2,\ldots$ and φ_o arbitrary.

Setting $n = 0$ in (A3-7) gives

$$\varphi_1 = f + \lambda\mathcal{K}\,\varphi_o \tag{A3-8}$$

and, setting $n = 1$,

$$\varphi_2 = f + \lambda\mathcal{K}\,\varphi_1 = f + \lambda\mathcal{K}\left[f + \lambda\mathcal{K}\,\varphi_o\right]$$

or

$$\varphi_2 = f + \lambda\mathcal{K}\,f + \lambda^2\mathcal{K}(\mathcal{K}\,\varphi_o)\ . \tag{A3-9}$$

Now

$$\mathcal{K}(\mathcal{K}\,\varphi_o) = \int_a^b G(t,\tau)\left\{\int_a^b G(\tau,\zeta)\,\varphi(\zeta)\,d\zeta\right\}d\tau$$

or, interchanging the order of integration,

$$\mathcal{K}(\mathcal{K}\,\varphi_o) = \int_a^b\left\{\int_a^b G(t,\tau)\,G(\tau,\zeta)\,d\tau\right\}\varphi_o(\zeta)\,d\zeta\ . \tag{A3-10}$$

Two conventions are now defined:

(i) $$\mathcal{H}^2\,\varphi(t) \triangleq \mathcal{H}(\mathcal{H}\,\varphi(t)) \tag{A3-11}$$

(ii) $$G_2(t,\tau) \triangleq \int_a^b G(t,\zeta)\; G(\zeta,\tau)\; d\zeta \tag{A3-12}$$

so that (A3-10) may be written

$$\mathcal{H}^2\,\varphi_o = \int_a^b G_2(t,\tau)\;\varphi_o(\tau)\;d\tau\ . \tag{A3-13}$$

Thus (A3-9) becomes

$$\varphi_2 = f + \lambda\mathcal{H}f + \lambda^2\mathcal{H}^2\varphi_o. \tag{A3-14}$$

Similarly, for $n = 2$, the third approximation is

$$\varphi_3 = f + \lambda\mathcal{H}f + \lambda^2\mathcal{H}^2 f + \lambda^3\mathcal{H}^3\varphi_o \tag{A3-15}$$

where

$$\mathcal{H}^3\varphi_o \triangleq \mathcal{H}(\mathcal{H}^2\varphi_o)\ . \tag{A3-16}$$

Thus

$$\begin{aligned}\mathcal{H}^3\varphi_o &= \int_a^b G(t,\tau)\left\{\int_a^b G_2(\tau,\zeta)\varphi_o(\zeta)d\zeta\right\}d\tau \\ &= \int_a^b\left\{\int_a^b G(t,\tau)\;G_2(\tau,\zeta)\;d\tau\right\}\varphi_o(\zeta)d\zeta\end{aligned} \tag{A3-17}$$

and

$$\mathcal{K}^3\varphi_o = \int_a^b G_3(t,\zeta)\ \varphi_o(\zeta)\ d\zeta \tag{A3-17}$$

where

$$G_3(t,\tau) \triangleq \int_a^b G(t,\zeta)\ G_2(\zeta,\tau)\ d\zeta\ . \tag{A3-18}$$

In this fashion the general notation is adopted

$$\mathcal{K}^n\ \varphi \triangleq \mathcal{K}(\mathcal{K}^{n-1}\varphi) \tag{A3-19}$$

and

$$G_n(t,\tau) \triangleq \int_a^b G(t,\zeta)\ G_{n-1}(\zeta,\tau)\ d\zeta \tag{A3-20}$$

with $G_1(t,\tau) \triangleq G(t,\tau)$.

Clearly,

$$\mathcal{K}(\mathcal{K}^n\varphi) = \mathcal{K}^n(\mathcal{K}\varphi) \tag{A3-21}$$

and

$$\mathcal{K}^j(\mathcal{K}^{n-j}\varphi) = \mathcal{K}^{n-j}(\mathcal{K}^j\varphi) = \mathcal{K}^n\varphi \tag{A3-22}$$

for $n = 1,2,\ldots$ and $j = 1,2,\ldots, n-1$.

Combining (A3-19) and (A3-20) gives

$$\mathcal{K}^n\ \varphi(t) \triangleq \int_a^b G_n(t,\tau)\ \varphi(\tau)\ d\tau \tag{A3-23}$$

where $G_n(t,\tau)$ is called the <u>n^{th} iterated kernel of $G(t,\tau)$</u> and $\mathcal{K}^n$ is called the $\underline{n^{th}}$ degree of the Fredholm operator $\mathcal{H}$, equation (A3-5), and is by (A3-23) itself a Fredholm operator.

It is also clear from (A3-22) that

$$G_n(t,\tau) = \int_a^b G_j(t,\zeta)\, G_{n-j}(\zeta,\tau)\, d\zeta \qquad \text{(A3-24)}$$

for $n = 1,2,\ldots$ and $j = 1,2,\ldots,n-1$.

Using this compact notation, the successive substitution of the preceding n approximations in equation (A3-7) yields

$$\varphi_{n+1} = f + \lambda\mathcal{H} f + \lambda^2\mathcal{H}^2 f + \ldots + \lambda^n\mathcal{H}^n f + \lambda^{n+1}\mathcal{H}^{n+1}\varphi_o , \qquad \text{(A3-25)}$$

or,

$$\varphi_{n+1} = f + \sum_{i=1}^{n} \lambda^i\mathcal{H}^i f + \lambda^{n+1}\mathcal{H}^{n+1}\varphi_o . \qquad \text{(A3-26)}$$

Equations (A3-25) and (A3-26) define a sequence of successive approximations $\{\varphi_n(t)\}$ which, under the proper hypotheses, converges to the solution of (A3-2) and hence (A3-1).

A proof of the convergence of (A3-26) will now be given and estimates of the rate of convergence derived. Following this derivation several well known results and theorems will be listed, without proofs, concerning the convergence of the sequence of successive approximations.

<u>Theorem (A3-I)</u>

The sequence of successive approximations (A3-26) converges uniformly and absolutely to the unique absolutely integrable solution of the Volterra equation (A3-1) for all $t \in [a,b]$ if the following hypotheses hold:

(i) The kernel $K(t,\tau)$ is bounded,

$$|K(t,\tau)| \leq M \quad \text{for} \quad (t,\tau) \in [a,b] \times [a,b], \tag{A3-27}$$

(ii) The function $f(t)$ is absolutely integrable on $[a,b]$

$$\int_a^b |f(\tau)| d\tau \stackrel{\Delta}{=} A < \infty, \tag{A3-28}$$

(iii) The initial approximation, $\varphi_o(t)$, is absolutely integrable on $[a,b]$

$$\int_a^b |\varphi(\tau)| \, d\tau \stackrel{\Delta}{=} B < \infty . \tag{A3-29}$$

<u>Proof</u>: Since $G(t,\tau)$ is defined by (A3-3), it follows that

$$G_2(t,\tau) = \begin{cases} \int_\tau^t K(t,\zeta)K(\zeta,\tau)d\zeta & \text{for} \quad a \leq \tau \leq t \leq b \\ 0 & \text{for} \quad a \leq t < \tau \leq b \end{cases} \tag{A3-30}$$

and, by induction,

$$G_n(t,\tau) = \begin{cases} \int_\tau^t K_{n-j}(t,\zeta)K_j(\zeta,\tau)d\zeta & \text{for} \quad a \leq \tau \leq t \leq b \\ 0 & \text{for} \quad a \leq t < \tau \leq b \end{cases} \tag{A3-31}$$

for any $j = 1,2,\ldots,n-1$ and all $n = 2,3,\ldots$.

Using the bound on $K(t,\tau)$, it follows that

$$
\begin{aligned}
|G(t,\tau)| &\leq M\,, \\
|G_2(t,\tau)| &\leq \int_\tau^t |K(t,\zeta)|\;|K(\zeta,\tau)|d\zeta \\
&\leq M^2(t-\tau) \\
&\vdots \\
|G_n(t,\tau)| &\leq M^n \frac{(t-\tau)^{n-1}}{(n-1)!} \quad \text{for} \quad t \geq \tau
\end{aligned}
\tag{A3-32}
$$

and $n = 1,2,\ldots$.

Thus

$$
\begin{aligned}
|\mathcal{K}^j f(t)| &\leq \int_a^b |G_j(t,\tau)|\;|f(\tau)|d\tau = \\
&\int_a^t |K_j(t,\tau)|\;|f(\tau)|\,d\tau \\
&\leq M^j \frac{1}{(j-1)!} \int_a^t (t-\tau)^{j-1}\;|f(\tau)|\,d\tau\,.
\end{aligned}
\tag{A3-33}
$$

But, by Theorem (A1-X),

$$
\frac{1}{(j-1)!}\int_a^t (t-\tau)^{j-1}\;|f(\tau)|d\tau = \underbrace{\int_a^t \cdots \int_a^t}_{j \text{ times}} |f(\tau)|\,\underbrace{d\tau \ldots d\tau}_{j \text{ times}} \tag{A3-34}
$$

and, since

$$\int_a^t |f(\tau)|d\tau \leq A ,$$

$$\int_a^t \left\{\int_a^{\tau} |f(\zeta)|\, d\zeta\right\} d\tau \leq A(t-a)$$

$$\int_a^t \left\{\int_a^{t_3} \int_a^{t_2} |f(t_1)|dt_1 dt_2\right\} dt_3 \leq A \int_a^t (t_3-a)\, dt_3$$

$$\leq A \frac{(t-a)^2}{2}$$

$$\vdots$$

$$\underbrace{\int_a^t \cdots \int_a^t}_{j \text{ times}} |f(\tau)| \underbrace{d\tau \ldots d\tau}_{j \text{ times}} \leq A \frac{(t-a)^{j-1}}{(j-1)!} . \qquad \text{(A3-35)}$$

The bound (A3-33) becomes

$$|\mathcal{H}^j f(t)| \leq AM^j \frac{(t-a)^{j-1}}{(j-1)!} . \qquad \text{(A3-36)}$$

Since

$$|\lambda^j \mathcal{H}^j f(t)| \leq A|\lambda|^j M^j \frac{(t-a)^{j-1}}{(j-1)!} =$$

$$\leq A|\lambda|M \frac{\{|\lambda|M(t-a)\}^{j-1}}{(j-1)!} , \qquad \text{(A3-37)}$$

$$|\lambda^j \mathcal{H}^j f(t)| \leq A|\lambda|M \frac{\{|\lambda|M (b-a)\}^{j-1}}{(j-1)!} \qquad \text{(A3-38)}$$

for all t in $[a,b]$.

Thus, the infinite series

$$\sum_{j=1}^{\infty} \lambda^j \mathcal{H}^j f(t) \tag{A3-39}$$

converges uniformly and absolutely, for all λ, on the interval $[a,b]$ since,

$$\left| \sum_{j=1}^{\infty} \lambda^j \mathcal{H}^j f(t) \right| \leq \sum_{j=1}^{\infty} |\lambda^j \mathcal{H}^j f(t)|$$

$$\leq A|\lambda|M \sum_{j=1}^{\infty} \frac{\{|\lambda|M(b-a)\}^{j-1}}{(j-1)!} =$$

$$\leq A|\lambda|M e^{\{|\lambda|M(b-a)\}} \quad . \tag{A3-40}$$

Also, from (A3-37) and (A3-29),

$$|\lambda^{n+1} \mathcal{H}^{n+1} \varphi_o(t)| \leq B|\lambda|M \frac{\{|\lambda|M(t-a)\}^n}{n!} \quad . \tag{A3-41}$$

Since the right side of (A3-41) is the general term of the convergent exponential series, it follows that it tends to zero as n tends to infinity.

Thus,

$$\lim_{n \to \infty} |\lambda^{n+1} \mathcal{H}^{n+1} \varphi_o(t)| = 0 \, . \tag{A3-42}$$

Passage to the limit is now justified in equation (A3-26).

Letting

$$\varphi^*(t) \triangleq \lim_{n \to \infty} \varphi_{n+1}(t) \quad , \tag{A3-43}$$

$$\varphi^*(t) = f(t) + \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^j f(t) , \tag{A3-44}$$

the series (A3-44) converging uniformly and absolutely on $[a,b]$.

To see that $\varphi^*(t)$ is a solution of equation (A3-1) consider $f + \lambda \mathcal{K} \varphi^*$.

$$\begin{aligned} \lambda \mathcal{K} \varphi^* &= \lambda \mathcal{K} \left\{ f + \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^j f \right\} \\ &= \lambda \mathcal{K} f + \lambda \mathcal{K} \left\{ \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^j f \right\}. \end{aligned} \tag{A3-45}$$

By uniformity,

$$\begin{aligned} \mathcal{K} \left\{ \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^j f \right\} &= \sum_{j=1}^{\infty} \mathcal{K} \left\{ \lambda^j \mathcal{K}^j f \right\} \\ &= \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^{j+1} f \end{aligned} \tag{A3-46}$$

so

$$\lambda \mathcal{K} \varphi^* = \lambda \mathcal{K} f + \sum_{j=1}^{\infty} \lambda^{j+1} \mathcal{K}^{j+1} f , \tag{A3-47}$$

or,

$$\lambda \mathcal{K} \varphi^* = \sum_{j=1}^{\infty} \lambda^j \mathcal{K}^j f . \tag{A3-48}$$

Thus

$$f + \lambda \mathcal{H} \varphi^* = f + \sum_{j=1}^{\infty} \lambda^j \mathcal{H}^j f \tag{A3-49}$$

and the right hand side is recognized by (A3-44) as φ^*.

Since

$$f + \lambda \mathcal{H} \varphi^* = \varphi^*$$

the function $\varphi^*(t)$ constructed by the series (A3-44) is a solution of (A3-2) and hence of (A3-1).

The solution $\varphi^*(t)$ is absolutely integrable since it is the sum of $f(t)$ and the series

$$\sum_{j=1}^{\infty} \lambda^j \mathcal{H}^j f(t)$$

which, by (A3-40), is bounded on $[a,b]$.

To show that φ^* is unique, suppose there were two absolutely integrable solutions of (A3-1)

$$r(t) \text{ and } s(t).$$

Their difference

$$\omega(t) = r(t) - s(t) \tag{A3-50}$$

satisfies the homogeneous equation

$$\omega(t) = \lambda \int_a^b G(t,\tau)\, \omega(\tau)\, d\tau \tag{A3-51}$$

which can be written in operator notation as

$$\omega(t) = \lambda \mathcal{K}\, \omega(t) \; . \tag{A3-52}$$

Substituting the right hand side (A3-52) for $\omega(t)$ in the same equation yields

$$\omega(t) = \lambda\mathcal{K}\,(\lambda\mathcal{K}\,\omega(t)) = \lambda^2\mathcal{K}^2\,\omega(t) \; .$$

Repeating this process n times,

$$\omega(t) = \lambda^n\mathcal{K}^n\,\omega(t) \tag{A3-53}$$

and since $\omega(t)$ is absolutely integrable, by (A3-50), the bound of (A3-37) shows that

$$|\omega(t)| \leq |\lambda| M \frac{\{|\lambda| M(t-a)\}^{n-1}}{(n-1)!} \int_a^b |\omega(\tau)|\, d\tau \; . \tag{A3-54}$$

Letting $n \rightarrow \infty$ in (A3-54) yields $|\omega(t)| \leq 0$ from which it follows that $r(t) = s(t)$ and there can be at most one absolutely integrable solution to (A3-1).

<u>Rate of Convergence and Error Bounds</u>

To estimate the rate of convergence of the sequence of successive approximations $\{\varphi_n(t)\}$ to $\varphi^*(t)$, consider the truncation error $\{\varphi^*(t) - \varphi_{n+1}(t)\}$.

Substituting for φ^* and φ_{n+1},

$$\varphi^* - \varphi_{n+1} = \left(f + \sum_{j=1}^{\infty} \lambda^j \mathscr{K}^j f\right) - \left(f + \sum_{i=1}^{\infty} \lambda^i \mathscr{K}^i f + \lambda^{n+1} \mathscr{K}^{n+1} \varphi_o\right),$$

$$\varphi^* - \varphi_{n+1} = \sum_{j=n+1}^{\infty} \lambda^j \mathscr{K}^j f - \lambda^{n+1} \mathscr{K}^{n+1} \varphi_o \;. \tag{A3-55}$$

Thus

$$|\varphi^* - \varphi_{n+1}| \leq \sum_{j=n+1}^{\infty} |\lambda^j \mathscr{K}^j f| + |\lambda^{n+1} \mathscr{K}^{n+1} \varphi_o| \;. \tag{A3-56}$$

By (A3-37),

$$\sum_{j=n+1}^{\infty} |\lambda^j \mathscr{K}^j f| \leq \sum_{j=n+1}^{\infty} A|\lambda|M \frac{\{|\lambda|M(t-a)\}^{j-1}}{(j-1)!} =$$

$$\leq A|\lambda|M \left\{ \sum_{j=1}^{\infty} \frac{\{|\lambda|M(t-a)\}^{j-1}}{(j-1)!} - \sum_{j=1}^{n} \frac{\{|\lambda|M(t-a)\}^{j-1}}{(j-1)!} \right\}$$

$$\leq A|\lambda|M \left\{ e^{\{|\lambda|M(t-a)\}} - \sum_{j=o}^{n-1} \frac{\{|\lambda|M(t-a)\}^j}{j!} \right\} \tag{A3-57}$$

and with this result and (A3-41), (A3-56) becomes

$$|\varphi^*(t) - \varphi_{n+1}(t)| \leq A|\lambda|M \left\{ e^{\{|\lambda|M(t-a)\}} - \sum_{j=o}^{n-1} \frac{\{|\lambda|M(t-a)\}^j}{j!} \right\}$$

$$+ B|\lambda|M \frac{\{|\lambda|M(t-a)\}^n}{n!} \tag{A3-57}$$

for all t in $[a,b]$.

The inequality (A3-57) provides an upper bound for the truncation error in approximating the solution $\varphi^*(t)$ by $\varphi_{n+1}(t)$.

A Conventional Initial Approximation

Since the sequence of successive approximations converges for an arbitrary initial approximation $\varphi_o(t)$ it is customary to begin the iteration by choosing $\varphi_o(t) = f(t)$ unless advance information about the nature of the solution is available.

For this choice of φ_o, the iteration scheme becomes

$$\varphi_{n+1} = f + \sum_{j=1}^{n+1} \lambda^i \mathcal{K}^i f \tag{A3-58}$$

and a sharper bound on the truncation error is possible.

Letting $\varphi_o = f$,

$$\left\{\varphi^* - \varphi_{n+1}\right\} = \sum_{j=n+2}^{\infty} \lambda^j \mathcal{K}^j f \tag{A3-59}$$

and

$$|\varphi^* - \varphi_{n+1}| \leq \sum_{j=n+2}^{\infty} |\lambda^j \mathcal{K}^j f|$$

which, by (A3-57), yields

$$|\varphi^*(t) - \varphi_{n+1}(t)| \leq A|\lambda|M \left\{ e^{\{|\lambda|M(t-a)\}} - \sum_{j=o}^{n} \frac{\{|\lambda|M(t-a)\}^j}{j!} \right\} \tag{A3-60}$$

for all λ, and all t in $[a,b]$ when $\varphi_0(t) = f(t)$.

A Special Case of Theorem (A3-I)

A less general but useful set of hypotheses under which the results of Theorem (A3-I) hold are stated as

Theorem (A3-II)[44]

If in equation (A3-1) the kernel $K(t,\tau)$ is finite and its discontinuities, if it has any, are regularly distributed* in the triangular region T,

$$T = \left\{(t,\tau) \,:\, a \leq \tau \leq t \leq b\right\},$$

a necessary and sufficient condition that the sequence of successive approximations, (A3-26) converge absolutely and uniformly to the unique continuous solution on $[a,b]$ is that $f(t)$ and $\varphi_0(t)$ be continuous on $[a,b]$.

For this special case, since continuity on a compact** set implies boundedness, the following bounds hold:

(i) $|K(t,\tau)| \leq M$ on $[a,b] \times [a,b]$,

(ii) $|f(\tau)| \leq m$ on $[a,b]$, (A3-61)

(iii) $|\varphi_0(\tau)| \leq c$ on $[a,b]$. (A3-62)

Following the outline of the proof of Theorem (A3-I), estimate (A3-33) becomes

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*See Definition (A1-II).

**A set is compact if, and only if, it is closed and bounded. See Apostol, Ref. (22).

$$|\mathcal{K}^{j}\ f(t)| \leq m\ \frac{M^{j}(t-a)^{j}}{j!} \tag{A3-63}$$

and (A3-37) becomes

$$|\lambda^{j}\mathcal{K}^{j}\ f(t)| \leq m\ \frac{\{|\lambda|M(t-a)\}^{j}}{j!}. \tag{A3-64}$$

Similarly, (A3-41) becomes

$$\lambda^{n+1}\mathcal{K}^{n+1}\ \varphi_{o}(t)| \leq c\ \frac{\{|\lambda|M(t-a)\}^{n+1}}{(n+1)!}\ , \tag{A3-65}$$

and the error estimates are

$$|\varphi^{*}(t)\ -\ \varphi_{n+1}(t)| \leq m \left\{ e^{\{|\lambda|M(t-a)\}} - \sum_{j=o}^{n} \frac{\{|\lambda|M(t-a)\}^{j}}{j!} \right\}$$

$$+\ c\ \frac{\{|\lambda|M(t-a)\}^{n+1}}{(n+1)!} \tag{A3-66}$$

for $\varphi_{o}(t)$ arbitrary

and

$$|\varphi^{*}(t) - \varphi_{n+1}(t)| \leq m \left\{ e^{\{|\lambda|M(t-a)\}} - \sum_{j=o}^{n+1} \frac{\{|\lambda|M(t-a)\}^{j}}{j!} \right\} \tag{A3-67}$$

for $\varphi_{o}(t) = f(t)$.

<u>The Neumann Series and the Resolvent Kernel</u>

The series (A3-44) generated by the method of successive approximations is called the <u>Neumann series</u> and is usually written

$$\varphi^*(t) = f(t) + \sum_{j=1}^{\infty} \lambda^j \int_a^b G_j(t,\tau) f(\tau) d\tau . \tag{A3-68}$$

Interchanging limit processes, (A3-68) becomes

$$\varphi^*(t) = f(t) + \int_a^b \left\{ \sum_{j=1}^{\infty} \lambda^j \, G_j(t,\tau) \right\} f(\tau) d\tau \tag{A3-69}$$

or

$$\varphi^*(t) = f(t) + \lambda \int_a^b \Gamma(t,\tau;\lambda) \; f(\tau) d\tau \tag{A3-70}$$

where

$$\Gamma(t,\tau;\lambda) \triangleq \sum_{j=1}^{\infty} \lambda^{j-1} \, G_j(t,\tau). \tag{A3-71}$$

The function $\Gamma(t,\tau;\lambda)$ is called the resolvent kernel of equation (A3-2) and it is easily shown[45] that the following relationships hold:

(i) $$\Gamma(t,\tau;\lambda) = G(t,\tau) + \lambda \int_a^b G(t,\zeta)\, \Gamma(\zeta,\tau;\lambda)\, d\zeta , \tag{A3-72}$$

(ii) $$\Gamma(t,\tau;\lambda) = G(t,\tau) + \lambda \int_a^b \Gamma(t,\zeta;\lambda)\, G(\zeta,\tau) \; d\zeta . \tag{A3-73}$$

Equation (A3-70) shows that by calculating the resolvent kernel, by (A3-71), the solution φ^* may be written for arbitrary $f(t)$ whereas the solution by successive approximations, (A3-44), is valid only for the particular $f(t)$ with which the iteration is begun.

Square Integrability and Convergence in the Mean

Definition (A3-III)[46]

A sequence $\{\varphi_n(t)\}$ of square integrable functions on the interval $[a,b]$ is said to converge in the mean to the square integrable function $\varphi(t)$, called the limit in the mean of $\varphi_n(t)$, if

$$\lim_{n \to \infty} \left\{ \int_a^b \{\varphi(\tau) - \varphi_n(\tau)\}^2 \, d\tau \right\}^{1/2} = 0 .$$

Theorem (A3-II)[47]

The Neumann series (A3-44) for the Volterra equation (A2-1) is convergent in the mean for all λ if, and only if, the kernel $K(t,\tau)$ and the function $f(t)$ are square integrable.

This solution is unique up to a null function in the set of square integrable functions and is given by (A3-70) where the series (A3-71) converges almost everywhere.

The resolvent kernel also satisfies (A3-72) and (A3-73).

APPENDIX IV

SOME OTHER METHODS OF COMPUTING $r(t)$

Convolution Kernels and Resolvents

A Volterra kernel is said to be of the convolution type if

$$K(t,\tau) = k(t-\tau) \quad \text{for} \quad \tau \leq t . \tag{A4-1}$$

For convolution kernels the analysis of Appendix III, (A3-3), has

$$G(t,\tau) = \begin{cases} k(t-\tau) & \text{for} \quad 0 \leq \tau \leq t \leq b \\ 0 & \text{for} \quad 0 \leq t < \tau \leq b \end{cases} , \tag{A4-2}$$

and the iterated kernels $G_j(t,\tau)$ are easily seen to all be of convolution type.

In fact, substituting (A4-2) in (A3-30)

$$G_2(t,\tau) = \begin{cases} \int_\tau^t k(t-\zeta)\, k(\zeta-\tau)\, d\zeta & \text{for} \quad 0 \leq \tau \leq t \leq b \\ 0 & \text{for} \quad 0 \leq t < \tau \leq b \end{cases} \tag{A4-3}$$

and, changing variables, letting $\alpha = \zeta - \tau$,

$$G_2(t,\tau) = \begin{cases} \int_0^{(t-\tau)} k([t-\tau]-\alpha)\, k(\alpha)\, d\alpha & \text{for} \quad \tau \leq t \\ 0 & \text{if} \quad t < \tau \end{cases} . \tag{A4-4}$$

Using the notation of (2-18) and (A3-30),

$$k_2(t-\tau) = \begin{cases} \int_o^{(t-\tau)} k([t-\tau]-\zeta)\, k(\zeta)\, d\zeta \\ \qquad\qquad \text{for} \quad \tau \leq t \,, \\ 0 \qquad \text{for} \quad t < \tau \end{cases} \tag{A4-5}$$

and equation (A4-4) becomes

$$G_2(t,\tau) = k_2(t-\tau) \;. \tag{A4-6}$$

Repeating this process gives

$$G_n(t,\tau) = k_n(t-\tau) \tag{A4-7}$$

where

$$k_n(t-\tau) \triangleq \begin{cases} \int_o^{(t-\tau)} k_{n-j}([t-\tau]-\zeta)\, k_j(\zeta)\, d\zeta \\ \qquad\qquad \text{for} \quad \tau \leq t \,, \\ 0 \qquad \text{for} \quad t < \tau \,, \end{cases} \tag{A4-8}$$

for $n = 1,2,\ldots$ and any $j = 1,2,\ldots,n-1$,

and

$$k_1(t - \tau) \triangleq k(t-\tau) \;. \tag{A4-9}$$

It is simpler to write (A4-8) and (A4-9) as

$$k_n(t) \triangleq \begin{cases} \int_o^t k_{n-j}(t-\tau)\, k_j(\tau)\, d\tau \\ \qquad\qquad \text{for} \quad t \geq 0 , \\ 0 \quad \text{for} \quad t < 0 \end{cases} \tag{A4-10}$$

for $n = 1,2,...$ and any $j = 1,2,...,n-1,$

and

$$k_1(t) \triangleq k(t) \tag{A4-11}$$

which reveals that the n^{th} iterated kernel $G_n(t,\tau)$ of Appendix III becomes the n-fold self-convolution for kernels of the convolution type.

Since the iterated kernels are all of convolution type, the resolvent kernel, (A3-71) is also a convolution kernel and is written

$$\Gamma(t,\tau;\lambda) \triangleq \begin{cases} R(t-\tau; \lambda) & \text{for} \quad \tau \leq t \\ 0 & \text{for} \quad t < \tau \end{cases} \tag{A4-12}$$

and the series (A3-71) becomes

$$R(t;\lambda) \triangleq \sum_{j=1}^{\infty} \lambda^{j-1}\, k_j(t) \ . \tag{A4-13}$$

The two relationships (A3-72) and (A3-73) become

(i) $$R(t;\lambda) = k(t) + \lambda \int_o^t k(t-\tau)\, R(\tau;\lambda)\, d\tau \tag{A4-14}$$

and

$$\text{(ii)} \quad R(t;\lambda) = k(t) + \lambda \int_0^t R(t-\tau;\lambda)\, k(\tau)\, d\tau \tag{A4-15}$$

for convolution kernels.

Some Results of G. C. Evans

In the division algorithm of Chapter II the function $r(t)$, (2-38), is seen to be the resolvent kernel, (A4-13), of $k(t) = x'(t)$ and is shown in (2-48) to satisfy the equation

$$r(t) = k(t) + \lambda \int_0^t k(t-\tau)\, r(\tau)\, d\tau \tag{A4-16}$$

provided

$$\lambda \triangleq -\frac{1}{x(o)} . \tag{A4-17}$$

G. C. Evans* showed for integral equations of the type (A4-16) that if $k(t)$ satisfies the differential equation

$$L\, k(t) = 0 , \tag{A4-18}$$

where is the linear differential operator

$$L \triangleq a_o(t) + a_1(t) \frac{d}{dt} + \ldots + a_n(t) \frac{d^n}{dt^n} , \tag{A4-19}$$

then it is possible to determine the corresponding differential equation

$$\tilde{L}\, r(t) = 0 , \tag{A4-20}$$

*Ref. (14).

with

$$\tilde{L} \triangleq b_o(t) + b_1(t)\,\frac{d}{dt} + \ldots + b_n(t)\,\frac{d^n}{dt^n}\,, \tag{A4-21}$$

satisfied by $r(t)$.

To find $\tilde{L}$, operate with L on $r(t)$ as given by (A4-16)

$$Lr = \cancelto{0}{Lk} + \lambda L \left\{ \int_o^t k(t-\tau)\, r(\tau)\, d\tau \right\} \tag{A4-22}$$

recognizing that $Lk = 0$.

Now

$$\begin{aligned} L\left\{ \int_o^t k(t-\tau)\, r(\tau)\, d\tau \right\} = a_o(t) & \int_o^t k(t-\tau)\, r(\tau)\, d\tau \; + \\ & + a_1(t)\,\frac{d}{dt}\int_o^t k(t-\tau)\, r(\tau)\, d\tau \; + \\ & \;\;\vdots \\ & + a_n(t)\,\frac{d^n}{dt^n}\int_o^t k(t-\tau)\, r(\tau)\, d\tau\,. \end{aligned} \tag{A4-23}$$

The differentiations of (A4-23) become, by Theorem (A1-II),

$$\frac{d}{dt}\int_o^t k(t-\tau)\, r(\tau)\, d\tau = \int_o^t k'(t-\tau)\, r(\tau)\, d\tau + k(o)\, r(t) \tag{A4-24}$$

$$\frac{d^2}{dt^2}\int_o^t k(t-\tau)\, r(\tau)\, d\tau = \int_o^t k''(t-\tau)\, r(\tau)\, d\tau + k'(o)r(t) + k(o)r'(t) \tag{A4-25}$$

and

$$\frac{d^n}{dt^n}\int_o^t k(t-\tau)r(\tau)d\tau = \int_o^t k^{(n)}(t-\tau)r(\tau)d\tau + \sum_{j=o}^{n-1} k^{(n-j)}(o)\, r^{(j)}(t) \quad \text{(A4-26)}$$

for $n = 1,2,\ldots$.

Thus

$$\begin{aligned} L\left\{\int_o^t k(t-\tau)\, r(\tau)\, d\tau\right\} = {} & a_o(t)\int_o^t k(t-\tau)\, r(\tau)\, d\tau \; + \\ & + a_1(t)\left[\int_o^t k'(t-\tau)r(\tau)d\tau + k(o)\, r(t)\right] + \\ & + a_2(t)\left[\int_o^t k^{(2)}(t-\tau)r(\tau)d\tau + k'(o)r(t) + k(o)r'(t)\right] + \\ & \;\vdots \\ & + a_n(t)\left[\int_o^t k^{(n)}(t-\tau)r(\tau)d\tau + \sum_{j=o}^{n-1} k^{(n-j)}(o)r^{(j)}(t)\right]. \end{aligned}$$

(A4-27)

Substituting (A4-27) in (A4-22),

$$L\, r - \lambda\, L\left\{\int_o^t k(t-\tau)\; r(\tau)\; d\tau\right\} = 0 \qquad \text{(A4-28)}$$

becomes

$$0 = r(t)[a_o(t) - \lambda\{k(o)a_1(t) + k'(o)a_2(t) + \ldots + k^{(n-1)}(o)\ a_n(t)\}] +$$

$$+ r'(t)[a_1(t) - \lambda\{k(o)a_2(t) + k'(o)a_3(t) + \ldots + k^{(n-2)}(o)\ a_n(t)\}] +$$

$$+ r^{(2)}(t)[a_2(t) - \lambda\{k(o)a_3(t) + k'(o)a_4(t) + \ldots + k^{(n-3)}(o)\ a_n(t)\}] +$$

$$\vdots$$

$$r^{(n-1)}(t)\ [a_{n-1}(t) - \lambda\, k'(o)\, a_n(t)] + r^{(n)}(t)\ [a_n(t)]\ . \qquad \text{(A4-29)}$$

Since (A4-29) is in the form (A4-20) the coefficients $\{b_i(t)\}$ of $\tilde{L}$ are

$$b_o(t) = a_o(t) - \lambda \sum_{j=1}^{n} k^{(j-1)}(o)\ a_j(t), \qquad \text{(A4-30)}$$

$$b_1(t) = a_1(t) - \lambda \sum_{j=2}^{n} k^{(j-2)}(o)\ a_j(t), \qquad \text{(A4-31)}$$

$$b_2(t) = a_2(t) - \lambda \sum_{j=3}^{n} k^{(j-3)}(o)\ a_j(t)\ , \qquad \text{(A4-32)}$$

$$\vdots$$

$$b_{n-1}(t) = a_{n-1}(t) - \lambda\, k'(o)\, a_n(t), \qquad \text{(A4-33)}$$

and

$$b_n(t) = a_n(t)\ . \qquad \text{(A4-34)}$$

For the m^{th} coefficient of $\tilde{L}$, with

$$0 \leq m \leq n - 1 ,$$

$$b_m(t) = a_m(t) - \lambda \sum_{j=m+1}^{n} k^{(j-m-1)}(o)\ a_j(t) \qquad \text{(A4-35)}$$

or, equivalently,

$$b_m(t) = a_m(t) - \lambda \sum_{j=o}^{n-(m+1)} k^{j}(o)\ a_{j+m+1}(t) \ . \qquad \text{(A4-36)}$$

Thus the coefficients of the differential equation $\tilde{L}\, r = 0$ are determined by equations (A4-30) through (A4-36) from the coefficients $\{a_i(t)\}$ of the differential equation $L\, k = 0$.

To actually solve (A4-20) for $r(t)$ it is necessary to know the initial conditions $\{r^{(j)}(o)\}$ for $j = 0,1,\ldots,n-1$.

These initial data can be gotten by successively differentiating (A4-16):

$$r'(t) = k'(t) + \lambda \int_o^t k'(t-\tau)\, r(\tau)\, d\tau + \lambda\, k(o)\, r(t)$$

$$r''(t) = k''(t) + \lambda \int_o^t k''(t-\tau)\, r(\tau)\, d\tau + \lambda k'(o)\, r(t) + \lambda k(o)\, r'(t)$$

$$\vdots$$

$$r^{(m)}(t) = k^{(m)}(t) + \lambda \int_o^t k^{(m)}(t-\tau) r(\tau) d\tau + \lambda \sum_{j=o}^{m-1} k^{(m-1-j)}(o) r^{(j)}(t) \qquad \text{(A4-37)}$$

for $m = 1,2,\ldots,n$.

Hence,

$$r(o) = k(o) \tag{A4-38}$$

$$r'(o) = k'(o) + \lambda k(o)\, r(o) \tag{A4-39}$$

$$r''(o) = k''(o) + \lambda k'(o)\, r(o) + \lambda k(o) r'(o) \tag{A4-40}$$

$$\vdots$$

$$r^{(m)}(o) = k^{(m)}(o) + \lambda \sum_{j=o}^{m-1} k^{(m-1-j)}(o)\, r^{(j)}(o) \tag{A4-41}$$

for $m = 1,2,\ldots,n-1$,

and the initial conditions $\{r^{(i)}(o)\}$ are gotten recursively from (A4-38) through (A4-41), in terms of the initial values $\{k^{(i)}(o)\}$.

Two Illustrative Examples

(i) Consider the example of Chapter V where

$$x(t) = e^{-at} \tag{A4-42}$$

so that

$$k(t) = x'(t) = -a\, e^{-at} \tag{A4-43}$$

and

$$k'(t) = a^2\, e^{-at} .$$

The simplest differential equation which $k(t)$ satisfies is

$$Lk = [-a - 1 \tfrac{d}{dt}]\, k(t) = 0 \tag{A4-44}$$

since

$$-ak(t) - 1k'(t) = 0 .$$

Thus

$$a_o(t) = -a \tag{A4-45}$$

$$a_1(t) = -1 . \tag{A4-46}$$

From (A4-30) and (A4-31) ,

$$b_o(t) = a_o(t) - \lambda k(o)\ a_1(t) \tag{A4-47}$$

$$b_1(t) = a_1(t) \tag{A4-48}$$

so that

$$b_o(t) = -a - \lambda\ (-a)(-1) = -a\ (1 + \lambda) \tag{A4-49}$$

$$b_1(t) = -1 . \tag{A4-50}$$

The initial condition term is, from (A4-38),

$$r(o) = k(o) = -\ a . \tag{A4-51}$$

Thus, since

$$\lambda = -\frac{1}{x(o)} = -\frac{1}{1} = -1 , \tag{A4-52}$$

it follows that

$$b_o(t) = 0 \tag{A4-53}$$

and that $r(t)$ is the solution of the initial value problem

$$\begin{cases} r'(t) = 0 \\ r(o) = -a \end{cases} . \tag{A4-54}$$

The solution is

$$r(t) = -a . \tag{A4-55}$$

(ii) Consider

$$x(t) = A \sin \omega t + B \cos \omega t , \tag{A4-56}$$

$$B \neq 0 ,$$

so that

$$k(t) = \omega A \cos \omega t - \omega B \sin \omega t , \tag{A4-57}$$

$$k'(t) = \omega A^2 \sin \omega t - \omega^2 B \cos \omega t \tag{A4-58}$$

and

$$k''(t) = -\omega^2 k(t) \tag{A4-59}$$

or

$$\begin{cases} k''(t) + \omega^2 k(t) = 0 \\ k(o) = \omega A , \quad k'(o) = -\omega^2 B \end{cases} . \tag{A4-60}$$

Here

$$\begin{aligned} a_o(t) &= \omega^2 \\ a_1(t) &= 0 \\ a_2(t) &= 1 \end{aligned} \tag{A4-61}$$

so that

$$b_o(t) = a_o(t) - \lambda k(o)\ a_1(t) - \lambda k'(o)\ a_2(t)$$
$$b_1(t) = a_1(t) - \lambda k(o)\ a_2(t) \qquad \text{(A4-62)}$$
$$b_2(t) = a_2(t)$$

becomes

$$b_o(t) = \omega^2 + \lambda\omega^2 B = \omega^2(1 + \lambda B)$$
$$b_1(t) = -\lambda\omega A \qquad \text{(A4-63)}$$
$$b_2(t) = 1 \ .$$

The initial conditions for $r(t)$ are

$$\begin{aligned} r(o) &= k(o) = \omega A \\ r'(o) &= k'(o) + \lambda k(o)\ r(o) \\ &= -\omega^2 B + \lambda\ \omega^2\ A^2 \\ &= \omega^2\ (-B + \lambda A^2) \ . \end{aligned} \qquad \text{(A4-64)}$$

Thus $r(t)$ is the solution of

$$\begin{cases} \dfrac{d^2}{dt^2} r - \lambda\omega A \dfrac{dr}{dt} + \omega^2(1 + \lambda B)\ r(t) = 0 \\ r(o) = \omega A, \quad r'(o) = \omega^2(-B + \lambda A^2) \end{cases} \qquad \text{(A4-65)}$$

for a general parameter λ .

Since for the inversion algorithm of Chapter II, equation (A4-17) requires that

$$\lambda = -\frac{1}{x(o)} = -\frac{1}{B} \,, \tag{A4-66}$$

the initial value problem for $r(t)$, (A4-65), reduces to

$$\begin{cases} \dfrac{d^2r}{dt^2} + \dfrac{A}{B}\,\omega\,\dfrac{dr}{dt} = 0 & \text{(A4-67)} \\ r(o) = \omega A & \text{(A4-68)} \\ r'(o) = -\dfrac{\omega^2}{B}(A^2+B^2)\,. & \text{(A4-69)} \end{cases}$$

The solution is easily found to be

$$r(t) = \omega A - \frac{\omega}{A}\,(A^2 + B^2)\left\{1 - e^{-\frac{A}{B}\omega t}\right\} \tag{A4-70}$$

for $a \neq 0$,

and

$$r(t) = -\omega^2 B t \tag{A4-71}$$

for $A = 0$.

<u>Note</u>: Example (ii) serves two purposes. It illustrates the method of Evans for a useful case and it is both a generalization of and a correction to his results as they are quoted in Davis (48), page 18, "Example," and in Davis (20) page 487, problem 5. There are several misprints in his results on these pages. In the first reference, (48), for example, all of the solutions to the problems have omitted various terms containing the parameter λ. To indicate the nature of the misprints, the correct statement of problems one and two are:

1. If $K(z) = A$, show that $k(z) = A\, e^{\lambda A z}$,

and

2. If $K(z) = A^2 z$, show that $k(z) = \frac{A}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}\; A\, t)$.

Since none of these other cited results are applicable to the work of this thesis, the details of the corrections are not included, and it is only mentioned here to warn the reader. As they stand, the published results hold for the special case when $\lambda = 1$ and it is likely that Davis had this useful case in mind.

A Theorem of Davis

Davis* presents a method using a "resolvent generatrix" which because of its operational simplicity and close resemblance to Z transform operations is now briefly illustrated by a Theorem and an example.

Theorem (A4-I)

The resolvent kernel for the equation

$$\varphi(t) + \int_0^t k(t-\tau)\, \varphi(\tau)\, d\tau = f(t) \qquad \text{(A4-72)}$$

is formally equivalent to the expansion

$$r(t) = r(o) + r'(o)t + r''(o)\frac{t^2}{2!} + \ldots \qquad \text{(A4-73)}$$

where the $r^{(i)}(o)$ are determined from the equation

$$\left\{1 + k(o)z + k'(o)z^2 + k''(o)z^3 + \ldots\ldots\right\}^{-1} = 1 + r(o)z + r'(o)z^2 + r''(o)z^3 + \ldots\ldots \qquad \text{(A4-74)}$$

---

*Ref. 20.

<u>Example</u>:

Consider

$$k(t) = e^t \tag{A4-75}$$

so

$$k^{(i)}(o) = 1 \quad \text{for all} \quad i = 1,2,\dots \,. \tag{A4-76}$$

Since

$$\left\{1 + z + z^2 + \dots\right\}^{-1} = \left\{\frac{1}{1-z}\right\}^{-1} = 1-z \,, \tag{A4-77}$$

it follows from

$$\left\{1 + z + z^2 + \dots\right\}^{-1} = 1 + r(o)z + r'(o)z^2 + r''(o)z^3 + \dots \tag{A4-78}$$

that

$$r(o) = -1 \,, \tag{A4-79}$$

$$r^{(j)}(o) = 0 \quad \text{for} \quad j = 1,2,\dots \,. \tag{A4-80}$$

Thus

$$r(t) = -1 \,. \tag{A4-81}$$

It should be pointed out that λ must be understood to be incorporated in the kernel of equation (A4-72) to use Theorem (A4-I) in a problem with $\lambda \neq 1$.

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After receiving the M.S.E.E. in 1962 from Stanford he joined the faculty of San Jose State College as an Assistant Professor of electrical engineering. He has been a summer staff member for both the Radio Corporation of America and the Mitre Corporation and was an instructor at Fairleigh Dickinson University.

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